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TWO DETERMINANTS

BY

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Let $a_1, a_2, \ldots \ldots$ and $d_1, d_2, \ldots \ldots$ be two sets of numbers, and denote the product $(a_r + d_1)(a_r + d_2) \ldots \ldots (a_r + d_s)$ by $a_r^{(s)}$. Since

$$a_r^{(s+1)} = a_r \cdot a_r^{(s)} + d_{s+1} \cdot a_r^{(s)},$$

where d_{s+1} is independent of r, the determinant

$$\begin{vmatrix} 1 & a_1^{(1)} & a_1^{(2)} & \cdot & \cdot & \cdot & a_1^{(n-1)} \\ 1 & a_2^{(1)} & a_2^{(2)} & \cdot & \cdot & \cdot & a_2^{(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_n^{(1)} & a_n^{(2)} & \cdot & \cdot & \cdot & a_n^{(n-1)} \end{vmatrix}$$

is independent of $d_1, d_2, \cdot \cdot \cdot$; to take $d_1, d_2, \cdot \cdot$ all zero is to replace $a_r^{(s)}$ by the power a_r^s, and the value of the determinant is therefore $\zeta^{1/2}(a_1 a_2 \ldots a_n)$, the product of all the differences $a_r - a_q$ with $r > q$.

A good many elementary determinants involving factorials are of this form, slightly disguised by the presence of common factors in rows and columns. For example, since

$$\zeta^{1/2}(0, 1, 2, \ldots n) = 1!\, 2!\, 3! \ldots n!$$

we have immediately

$$\begin{vmatrix} 1/m! & 1/(m+1)! & \cdot & \cdot & 1/(m+n)! \\ 1/(m+1)! & 1/(m+2)! & \cdot & \cdot & 1/(m+n+1)! \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1/(m+n)! & 1/(m+n+1)! & \cdot & \cdot & 1/(m+2n)! \end{vmatrix} = \frac{1!\, 2!\, 3! \ldots n!}{(m+n)!\, (m+n+1)! \ldots (m+2n)!}$$

We have a more interesting problem if there is, so to speak, an absent column, and the typical row of the determinant is

$$[1, a_r^{(1)}, a_r^{(2)}, \ldots a_r^{(q-2)}, a_r^{(q)}, a_r^{(q+1)}, a_r^{(q+2)}, \ldots a_r^{(n)}].$$

As before, this is reducible, but now only to

$$[1, a_r, a_r^2, \ldots a_r^{q-2}, a_r^{(q)}, a_r \cdot a_r^{(q)}, a_r^2 \cdot a_r^{(q)} \ldots a_r^{n-q} \cdot a_r^{(q)}].$$

The determinant, which we will denote by F_q, is independent of $d_{q+1}, d_{q+2}, \ldots d_n$, but not of $d_1, d_2, \ldots d_q$, of which it is a symmetric function.

Since

$$d_q - d_{q-1} = (a_r + d_q) - (a_r + d_{q-1})$$

we have identically

$$(d_q - d_{q-1}) a_r^{(q-2)} = (a_r + d_1)(a_r + d_2) \cdots (a_r + d_{q-2})(a_r + d_q)$$
$$- (a_r + d_1)(a_r + d_2) \cdots (a_r + d_{q-2})(a_r + d_{q-1})$$

and F_q is expressible as $(F_{q-1} - G_{q-1})/(d_q - d_{q-1})$, where F_{q-1} and G_{q-1} are determinants of the same kind as F_q, with the difference that the jump in degree is from column $q-2$ to column $q-1$ instead of from column $q-1$ to column q. We can repeat this operation of moving the jump leftwards, not merely until we have determinants in which the typical row begins $[1, (a_r + d_h)(a_r + d_k), \ldots]$, but until we have determinants in which the typical row begins $[(a_r + d_k), (a_r + d_h)(a_r + d_k), \ldots]$. It follows that F_q is of the form

$$\sum_k \gamma_k P_k$$

where P_k is the determinant whose typical row is

$$[(a_r + d_k), a_r(a_r + d_k), a_r^2(a_r + d_k), \ldots a_r^{n-1}(a_r + d_k)]$$

and γ_k depends only on $d_1, d_2, \ldots d_q$.

The value of P_k is

$$\zeta^{1/2}(a_1 a_2 \ldots a_n) \times \prod_r (a_r + d_k).$$

In the decomposition of F_q, G_{q-1} is independent of d_q and F_{q-1} is independent of d_{q-1}; at the next stage, F_{q-1} is expressed as

$$(F_{q-2} - G_{q-2}) / (d_q - d_{q-2}),$$

where G_{q-2} is independent of d_q and F_{q-2} is independent of d_{q-2}. Thus the coefficient γ_q of the particular determinant P_q in the expression for F_q is $1/\Pi'(d_q - d_p)$, where the product is formed for all the values $1, 2, \ldots q$ of p except q itself, and from the symmetry of F_q in $d_1, d_2, \ldots d_q$ we have explicitly

$$F_q = \zeta^{1/2}(a_1 a_2 \ldots a_n) \times \sum_{k=1}^{q} \left\{ \prod_{r=1}^{n} (a_r + d_k) / \prod_{p=1}^{q}{}' (d_k - d_p) \right\}.$$

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We can modify this expression in two ways. On the one hand, we have

$$F_q = \frac{\zeta^{1/2}(a_1 a_2 \ldots a_n)}{\zeta^{1/2}(d_1 d_2 \ldots d_q)} \begin{vmatrix} 1 & d_1 & d_1^2 & \ldots & d_1^{q-2} & \Pi(a_r + d_1) \\ 1 & d_2 & d_2^2 & \ldots & d_2^{q-2} & \Pi(a_r + d_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & d_q & d_q^2 & \ldots & d_q^{q-2} & \Pi(a_r + d_q) \end{vmatrix}$$

each of the products in the last column being formed for the values $1, 2, \ldots n$ of r. On the other hand, if $A_1, A_2, \ldots A_n$ are the elementary symmetric functions of $a_1, a_2, \ldots a_n$, then

$$\Pi(a_r + d_k) = A_n + A_{n-1} d_k + A_{n-2} d_k^2 + \cdots + d_k^n,$$

and if $h > q-1$ we recognise in

$$\sum_k d_k^h / \Pi'(d_k - d_p)$$

and in the quotient by $\zeta^{1/2}(d_1 d_2 \ldots d_q)$ of the determinant whose typical row is

$$[1, d_k, d_k^2, \ldots d_k^{q-2}, d_k^h]$$

two familiar expressions for H_{h-q-1}, where H_s is the sum of the homogeneous products of degree s that can be formed from $d_1, d_2, \ldots d_q$. Thus finally

$$F_q = \zeta^{1/2}(a_1 a_2 \cdots a_n)\{A_{n-q+1} + A_{n-q} H_1 + A_{n-q-1} H_2 + \cdots + A_n H_{q-1}\}.$$

ON THE LINES OF STRICTION OF A QUADRIC

BY

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Starting with the projective definition of the line of striction corresponding to a regulus λ of a quadric, it has been shown analytically that the line of striction is a rational quartic curve determining a (3, 1) correspondence between the generators of the two reguli λ and μ respectively.* Recently K. K. Mitra has proved the same result by synthetic methods.† The object of this note is to get the same result by apolarity methods. The method adopted has the advantage of giving an invariant specification of the lines of striction, and the associated (3, 1) correspondences.

The line of striction corresponding to a regulus of a quadric is the locus of the central point on the lines of the regulus. Metrically

* Baker. *Principles of Geometry*, Vol. III, 84-85.

† K. K. Mitra. *Bulletin of the Calcutta Math. Soc.*

the central point C on any generator may be defined as the point of intersection of the generator, with the line of shortest distance between this generator and a consecutive one. Projectively it may be defined as follows. Let q be the section of the quadric by the plane at infinity, and Ω the circle at infinity. Let a generator λ meet q at P, and R be the pole, with respect to Ω, of the tangent at P to q. If PR meets q at S, then the μ-generator through S, intersects the λ generator through P at the central point C.

Any point on the quadric, can be specified by its minimal co-ordinates (i.e.) the parameters λ and μ of the two generators which pass through the point. Let us choose the parametric systems, so that the parameters are equal for every point on q. This is always possible in the case of a central quadric, and the common value of the parameter may as well be taken to be the parameter of the point as one lying on the conic q.

If the binary quartic a_x^4 gives parametrically the points of contact with q of the common tangents of q and Ω, the (2, 2) correspondence between the tangents of q at the points x and y, which are conjugate with respect to Ω is given by

$$a_x^2 \ a_y^2 + \mathrm{K}(xy)^2 = 0 \qquad (1{\cdot}1)$$

If P be the point λ, tangents of q conjugate to the tangent at P are given parametrically by the quadratic

$$a_\lambda^2 a_y^2 + \mathrm{K}(\lambda y)^2 = 0 \qquad (1{\cdot}2)$$

so that the pole R of the tangent at P with respect to Ω, determines with respect to q the quadratic (1·2). If S be the point μ, so that PS passes through R, then the quadratic $(\lambda y)(\mu y)$ must be apolar to (1·2)

$$\text{(i.e.)} \quad \{a_\lambda^2 a_y^2 + \mathrm{K}(\lambda y)^2, \ (\lambda y)(\mu y)\}^2 = 0$$

$$\text{(i.e.)} \quad a_\lambda^3 a_\mu = 0 \qquad (1{\cdot}3)$$

Since the central point C is the point of intersection of the λ-generator through P and the μ-generator through S, (λ, μ) are the minimal co-ordinates of C, and (1·3) is the equation of the line of striction in minimal co-ordinates. It is obvious from its form, that it is a rational quartic curve determining a (3, 1) correspondence between the generators of the two systems λ and μ.

Similarly the equation of the line of striction corresponding to the μ-system is $a_\lambda a_\mu^3 = 0$. Thus we have

If the minimal co-ordinates of every point on the section q of a central quadric by the plane at infinity are chosen to be equal to the parameter of the point regarded as one on the conic q, and if $a_x^4=0$ *gives parametrically the common tangents of q and the circle at infinity* Ω, *the lines of striction on the quadric are given by the double binary equations* $a_\lambda^3 a_\mu=0$, and $a_\mu^3 a_\lambda=0$, *in the minimal co-ordinates* (λ, μ).

We shall now obtain the points of intersection of the two lines of striction. If (λ, μ) be a point of intersection, the pair (λ, μ) satisfy both the equations $a_\lambda a_\mu^3=0$ and $a_\lambda^3 a_\mu=0$. Hence the points λ, and μ on q are mutually polar points with respect to the tetrad a_x^4. Any such point λ is easily seen to be a fixed point of the (1, 9) correspondence between any point λ and the polar points of its polar points. Hence there are 10 fixed points, of which 4 coincide with the points a_x^4, the corresponding points μ being identical with them. The other six form three pairs of mutually polar points. For, if A be a fixed point, of its three polar points, there is one B for which A is a polar point. Hence B also is a fixed point.

We shall now prove that the three pairs are given parametrically by the three mutually apolar quadratics p_x^2, q_x^2, r_x^2, whose product forms the sextic covariant of a_x^4. It is easily seen that the roots of any one of the quadratics say p_x^2, are mutually polar with respect to $(q_x^2)^2$ and $(r_x^2)^2$, and hence with respect to a_x^4 since it can be expressed as a linear combination of the first two quartics.

If L be the pole of A, B with respect to q, there are then three positions of L, which, from the above result, must form the vertices of the common self-conjugate triangle of q and Ω. Hence the generators of either system through the points of a pair A, B, intersect at the extremities of a principal axes of the quadric. Thus we have

*The two lines of striction on a central quadric have 10 points in common, six of which being the extremities of the principal axes of the quadric, the other four being the points of contact of the isotropic tangent planes of the quadric.**

While, thus, we have a metrical specification of the points on the curves defined by the correspondences $a_\lambda^3 a_\mu=0$, and $a_\lambda a_\mu^3=0$, it will be interesting to find a metrical specification for the points of the cyclic defined by the (2, 2) polar correspondence $a_\lambda^2 a_\mu^2=0$.

* In the case of a hyperboloid of one sheet, which is the only real ruled quadric the only real points of intersection are the extremities of the two principal axes which intersect the quadric. See Sommerville: *Analytical Geometry,* page 304.

ON CERTAIN REMARKABLE POINTS CONNECTED WITH A TRIANGLE *

BY

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1. **Notation.**—Let ABC be a triangle ($BC=a$, $CA=b$, $AB=c$, $a+b+c=2p$) for which the orientation ABCA may be taken as positive; h_a h_b h_c the lengths of the altitudes AA′, BB′, CC′; A_1 B_1 C_1 the mid-points of the sides, (O) the circumcircle of radius R; (O_9) the nine-point circle; (I), (I_a), (I_b), (I_c) the tritangent circles with radii r, r_a, r_b, r_c; G, H and K the centroid, orthocentre and the Lemoine point; S the area, θ the Brocard angle; BCC_aB_a, CAA_bC_b, ABB_cA_c and $BCC'_aB'_a$, $CAA'_bC'_b$, $ABB'_cA'_c$ the squares with centres α, β, γ and α' β' γ' constructed externally and internally on the sides BC, CA, AB; H′ the orthocentre of the pedal triangle A′B′C′; O_1, O_a, O_b, O_c the centres of the circles circumscribed to the triangles $I_aI_bI_c$, $I\,I_bI_c$, $I\,I_cI_a$, $I\,I_aI_b$.

The points $A_1B_1C_1$ are the projections of the points α and α', β and β', γ and γ' on the sides BC, CA, AB of the triangle. We denote by α_b and α_c, α'_b and α'_c the projections of the points α and α' on the sides CA and AB; by β_c and β_a, β'_c and β'_a those of the points β and β' on the sides AB and BC; by γ_a and γ_b, γ'_a and γ'_b those of γ and γ' on the sides BC and CA.

2. Applying the theorem regarding the square of the median to the triangles OIH, OI_aH, OI_bH, OI_cH we obtain the known relations

$$\left.\begin{aligned}HI^2 &= 2O_9I^2+\tfrac{1}{2}HO^2-OI^2=4R^2+2r^2-\tfrac{1}{2}(a^2+b^2+c^2)\\ HI_a{}^2 &= 4R^2+2r_a{}^2-\tfrac{1}{2}(a^2+b^2+c^2);\ \&\text{c. }\dagger\end{aligned}\right\}\ \dots \quad (1)$$

The same theorem for the triangles O_1IH, O_aI_aH, O_bI_bH, O_cI_cH gives, using (1) and well-known results,

$$HO_1{}^2=2HO^2+2OI^2-HI^2=2(9R^2-\Sigma a^2)+2R(R-2r)-(4R^2+2r^2-\tfrac{1}{2}\Sigma a^2) \quad (2)$$

Noting that $\Sigma a^2+r^2+\Sigma r_a{}^2=16R^2$; $\Sigma a^2=2(p^2-r^2-4Rr)$
$=2[(p-a)^2-r_a{}^2+4Rr_a]=2$ similar results and using these expressions for Σa^2 in (2), and its analogues for $HO_a{}^2$, $HO_b{}^2$, $HO_c{}^2$, we obtain,

$$\begin{aligned}HO_1{}^2&=r_a{}^2+r_b{}^2+r_c{}^2-p^2\\ HO_a{}^2&=r^2+r_b{}^2+r_c{}^2-(p-a)^2;\ \&\text{c. }\dagger\end{aligned} \quad \dots \quad (3)$$

* [Translated from the original in French—*Ed.*]

† [The "&c." in this and following formulae stand for two more results in the original obtained by a cyclical interchange of *a*, *b* and *c*.—*Ed.*]

Moreover taking account of the relations

$$p^2 = r_b r_c + r_c r_a + r_a r_b$$

$$(p-a)^2 = r_b r_c - r r_b - r r_c \text{ ; \&c.}$$

the equations (3) may be rewritten more elegantly thus :

$$HO_1^2 = \tfrac{1}{2}[(r_b - r_c)^2 + (r_c - r_a)^2 + (r_a - r_b)^2]$$

$$HO_a^2 = \tfrac{1}{2}[(r_b - r_c)^2 + (r_b - r)^2 + (r_c - r)^2] \text{ ; \&c.} \qquad \dots \quad (4)$$

3. The in- and ex-circles of radii r', r'_a, r'_b, r'_c of the pedal triangle A′B′C′ have respectively for centres the orthocentre H and the vertices A, B, C when the triangle ABC is *not right right angled.*

For fixing our ideas, let us suppose that the angles of the fundamental triangle ABC are all acute or that one of them is obtuse. If we write $B'C' = a'$ &c. $a' + b' + c' = 2p'$, $S' =$ area of A′B′C′, we know that

$$S' = |2S \cos A \cos B \cos C| \qquad \dots \quad (5)$$

The lengths $2p'$, $2(p'-a')$, $2(p'-b')$, $(p'-c')$ are equal respectively to

$$|a \cos A + b \cos B + c \cos C| = |4R \sin A \sin B \sin C|$$

$$|-a \cos A + b \cos B + c \cos C| = |4R \sin A \cos B \cos C| \text{\&c.} \dots \quad (6)$$

Similarly the radii r', r'_a, r'_b, r'_c are equal to the lengths

$$\left| 2R \cos A \cos B \cos C \right| , \left| \frac{S}{R} \cot A \right| , \left| \frac{S}{R} \cot B \right| , \left| \frac{S}{R} \cot C \right| \dots \quad (7)$$

4. When the triangle ABC is acute-angled,

$$r' = 2R \cos A \cos B \cos C \text{ ; } r'_a = \frac{S}{R} \cot A \text{ etc.} \qquad \dots \quad (8)$$

The first of the relations (3) applied to the triangle A′ B′ C′ becomes

$$H'O^2 = r'^2_a + r'^2_b + r'^2_c - p'^2 = \left(\frac{S}{R}\right)^2 (\cot^2 A + \cot^2 B + \cot^2 C - 1)$$

$$= \left(\frac{S}{R}\right)^2 [(\cot A + \cot B + \cot C)^2 - \sum 2 \cot B \cot C - 1]$$

$$= \left(\frac{S}{R}\right)^2 [(\cot A + \cot B + \cot C)^2 - 3]$$

$$= \left(\frac{S}{R}\right)^2 (\cot^2 \theta - 3), \qquad \dots \quad (9)$$

whence we deduce

$$H'O = \frac{S}{R} (\cot^2 \theta - 3)^{1/2} = p' (\cot^2 \theta - 3)^{1/2} \qquad \dots{}^*(10)$$

* These formulae have been obtained in a different manner ; vide *Mathesis* 1934, p. 88.

If the triangle ABC is *obtuse-angled* in, A for example, it suffices to substitute $p'-a'$ by p' in the preceding formulae thus obtaining

$$H'O = \frac{S}{R}(\cot^2\theta - 3)^{1/2} = (p' - a')(\cot^2\theta - 3) \qquad \dots (11)$$

5. Taking *any* triangle ABC, let us obtain the resultant of nine forces

$$(\overline{\alpha A_1}, \overline{\alpha\alpha_b}, \overline{\alpha\alpha_c}), (\overline{\beta\beta_a}, \overline{\beta B_1}, \overline{\beta\beta_c}), (\overline{\gamma\gamma_a}, \overline{\gamma\gamma_b}, \overline{\gamma C_1}) \qquad \dots (12)$$

acting three by three at the points α, β, γ.

Let B_o, C_o be the projections of B_c, C_b on BC. The angle ABB_c being right and AB equal to BB_c, the triangles $A'AB$, B_oBB_c are equal and we have in magnitude and sign

$$\gamma\gamma_a = \tfrac{1}{2}(AA' + B_cB_o) = \tfrac{1}{2}(AA' + BA');$$

$$\beta\beta_a = \tfrac{1}{2}(AA' + A'C).$$

Hence
$$\beta\beta_a + \gamma\gamma_a = AA' + \tfrac{1}{2}BC$$

$$\overline{\beta\beta_a} + \overline{\gamma\gamma_a} = \overline{AA'} + \overline{A_1\alpha}$$

$$\overline{\alpha A_1} + \overline{\beta\beta_a} + \overline{\gamma\gamma_a} = \overline{AA'}$$

On the other hand, the points A_1, β_α γ_α being the mid-points of B_0C_0, C_0A', $A'B_0$ we have also,

$$\gamma_a A_1 : A_1\beta\alpha = \tfrac{1}{2}A'C_0 : \tfrac{1}{2}B_0A' = \beta\beta_a : \gamma\gamma_a;$$

the point A_1 is thus on the line of action of the resultant of forces $\overline{\beta\beta_a}$, $\overline{\gamma\gamma_a}$ and as it is already along the line of action of $\overline{\alpha A_1}$, the three forces have a resultant equipollent to the perpendicular AA' of ABC and acting along the median OA_1 of BC.

It follows that the resultant of the nine forces (12) is also that of three forces $\overline{OX}$, $\overline{OY}$, $\overline{OZ}$ acting through the circumcentre O and equipollent to AA', BB', CC'.

The distances $|AB_1| = \frac{1}{2}b$, $|AC_1| = \frac{1}{2}c$ of A from the forces $\overline{OY}$, $\overline{OZ}$ being universely proportional to their intensities h_b, h_c, the resultant $\overline{OX'}$ passes through A.

Moreover the angles O, Y of the triangle OYX' being equal to B, A of ABC, the magnitude of $\overline{OX'}$ is

$$|OX'| = \left|\frac{X'Y \sin A}{\sin B}\right| = \frac{ah_c}{b} = \frac{2aS}{bc} = \frac{a^2}{2R}$$

which allows us to write

$$\overline{OY} + \overline{OZ} = \overline{OX'} = a^2.\ \overline{OA}/2R^2$$

From three such relations, the resultant $\overline{OT}$ of $\overline{OX}$, $\overline{OY}$, $\overline{OZ}$ is seen to be

$$\overline{OT} = (a^2 . \overline{OA} + b^2 . \overline{OB} + c^2 \overline{OC})/4R^2 = (a^2 + b^2 + c^2)\ OK/4R^2 \quad \dots (13)$$

and the square of its magnitude is

$$\overline{OT}^2 = (R^2 \sum a^4 + 2R^2 \sum b^2c^2 \cos 2A)/16R^2$$

$$= [(\sum a^2)^2 - 4 \sum b^2c^2 \sin^2 A]/16R^2.$$

But $\sum a^2 = 4S \cot \theta$ and $bc \sin A = 2S$; hence

$$|OT| = S\ (\cot^2 \theta - 3)^{1/2}/R$$

Similar reasoning will show that the resultant of the nine forces

$$(\overline{\alpha' A_1}, \overline{\alpha'\alpha'_b}, \overline{\alpha'\alpha'_c}) ; (\overline{\beta'\beta'_a}, \overline{\beta'\beta_1}, \overline{\beta'\beta'_c}), (\overline{\gamma'\gamma'_a}, \overline{\gamma'\gamma'_b}, \overline{\gamma' C_1}) \quad \dots (15)$$

is identical with that of the forces (12), namely $\overline{OT}$.

We may therefore enunciate the following theorem :

The 18 forces represented by the perpendiculars from the centres of the squares on the sides of a triangle ABC to the three sides, have for resultant the force

$$\overline{OU} = \overline{OK} \cdot (\sum a^2)/2R^2 = 2S \cdot \overline{OK} \cot \theta \ / \ R^2$$

whose magnitude is

$$\overline{OU} = 2S\ (\cot^2\theta - 3)^{\frac{1}{2}}/R \quad \dots (16)^*$$

We obtain the same result directly by observing that the forces such as $\overline{\alpha A_1}$, $\overline{\alpha' A_1}$ cancel each other and that forces like $\overline{\alpha\alpha_b}$, $\overline{\alpha'\alpha'_b}$ have a resultant equipollent to the altitude $\overline{BB'}$ of which the line of action passes through C.

6. For a triangle ABC *which is not right-angled,* the equivalence of (10) and (14) gives $H' \equiv T$, which allows us to enunciate the following theorem which is probably new :

Let (α, β, γ) *and* $(\alpha', \beta', \gamma')$ *be the centres of the squares constructed exteriorly and interiorly on the sides of a triangle ABC which is not right-angled, O the centre of the circumcircle, H′ the orthocentre of the pedal triangle A′B′C′. If we take the nine perpendiculars from* (α, β, γ) *or* $(\alpha'\beta'\gamma')$ *to the three sides of the triangle as representing nine forces, these have a resultant which is represented in magnitude and direction by the line segment* $\overline{OH'}$.

* V. Thébault: *Mathesis,* Supplement 1933.

7. When the triangle ABC is right-angled at A, for example, the resultant of the nine forces (12) or (15) is given by

$$\overline{OT} = (a^2 + b^2 + c^2)/4R^2 = 2a^2/a^2 = 2OK$$

The resultant of nine forces represented by the perpendiculars from $(\alpha\ \beta\ \gamma)$ or $(\alpha'\ \beta'\ \gamma')$ *to the three sides of the right-angled triangle ABC, is represented in magnitude and direction by the line segment OK′ joining the circum-centre O to its "symmetrique" K′ with respect to the Lemoine point K of ABC.*

The point K′ which is the reflection of the mid-point of the hypotenuse with respect to the mid-point K of the altitude AA′ is no longer the orthocentre of A′B′C′ which is the flat triangle AA′.

ON THE MOTION OF A PARTICLE OF VARYING MASS

BY

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1. When the mass of a particle is a function of time, which is continuous, and admits a first, continuous derivative satisfying a Lipschitz condition, the law of motion is expressed by the following system of differential equations :

$$(1{\cdot}1) \qquad \frac{d}{dt}\left(m\,\frac{dx}{dt}\right) = X, \quad \frac{d}{dt}\left(m\,\frac{dy}{dt}\right) = Y, \quad \frac{d}{dt}\left(m\,\frac{dz}{dt}\right) = Z.$$

These equations can be considered as the mathematical formulation of the fundamental law of Dynamics. Then the law of Newton expressed by the equations

$$(1{\cdot}2) \qquad m\,\frac{d^2x}{dt^2} = X, \quad m\,\frac{d^2y}{dt^2} = Y, \quad m\,\frac{d^2z}{dt^2} = Z,$$

are only a particular case. But I think that it is more satisfactory to state first the law of Newton and then to deduce (1·1) from (1·2) by a passage to the limit. Some text-books,* as a matter of fact, do their best to convince the student that there exists a deduction of that sort but their demonstrations fall short of rigour. The purpose

* No need to say that I apologize in advance if some demonstration has been published already. I have not read all the text-books, and I refer here only to English text-books. The particle of varying mass is no part of the syllabus in French Universities. I could not consult German text-books or syllabuses.

of the present paper is to develop a rigorous demonstration of the above statement.

It is manifest that we must first consider the motion of a particle from time t_0 to time t_1, then from time t_1 to time t_2, from time t_2 to time t_3, and so on, the mass being constant within any of these intervals and the equations (1·2) applying therefore between their boundaries. Then, when the particle passes from an interval to the next one, we must suppose that the mass all at once increases, so that there are discontinuities of the first kind of the velocity at the common boundaries of the intervals, according to the law of impulse.

Thus the path of the particle is a curve, representing a continuous, bounded function of t, whose first derivative, which is also bounded, admits a certain number of discontinuities of the first kind.

We must then consider the set of such paths, generated by successive subdivisions of the same interval of time into n parts, n tending to infinity so that the greatest subdivision tends to zero. Under certain general conditions there exists a limiting set which is a curve C admitting everywhere a tangent which varies continuously and this curve defines the extension of the law of Newton (1·2) to (1·1).

2. Let us consider a particle of mass m. If at time t there is a sudden increment or decrease Δm of that mass, then V being the velocity at time $t+0$ and v the velocity at time $t-0$, we have

(2·1) $$(m+\Delta m)V = mv$$

In what follows we consider only the case of a motion along a straight line, so that there is only one variable of position, x. The law of motion proved in this first case can be extended without difficulty to the case of more dimensions.

It is also easy to generalize the demonstration to the case of an unbounded interval of time or to the case of a velocity which is unbounded in an *open* interval.

3. Let us consider a linear field of forces X; this function is either a function of x only or a function of x and t. We consider also the differential equation

(3·1) $$\frac{d^2\xi}{dt^2} = X$$

We shall suppose that in the domain (D)

(3·2) $$x_0 \leqslant x \leqslant x + a; \qquad t_0 \leqslant t \leqslant T$$

this equation defines ξ as a function of t, which is continuous in the

closed domain, and admits derivatives of the first and second order which are also continuous in the same closed domain.*

4. *Limit of certain polygonal lines.*

Let $f(x, y)$ be a continuous function of the real variables x and y in the domain

$$(4\cdot1) \qquad \begin{array}{c} x_0 \leqslant x \leqslant x_0 + a \\ y_0 - b \leqslant y \leqslant y_0 + b \end{array}$$

Let moreover that function satisfy the Lipschitz condition

$$(4\cdot2) \qquad | f(x,y') - f(x,y'') | < K | y' - y'' |$$

in the domain (4·1).

K being a positive number. If M is the upper boundary of $|f(x, y)|$ in (4·1), the method of Cauchy-Lipschitz establishes the existence of a solution of the differential equation

$$(4\cdot3) \qquad \frac{dy}{dx} = f(x, y)$$

such that $y = y_0$ when $x = x_0$. This solution defines a curve C which is the limit of the polygonal line

$$(4\cdot4) \quad Y = y_0 + f(x_0,y_0)(x_1 - x_0) + f(x_1,y_1)(x_2 - x_1) + \ldots + f(x_{n-1},y_{n-1})(A - x_{n-1})$$

when n tends to infinity and the greatest of the subdivisions $(x_i - x_{i-1})$ tends to zero, A being the smaller of the numbers a and b/M.†

Let us now consider the following case: to each subdivision of the interval (x, A) into n parts corresponds a set of n functions $f_i(x,y)$, $(i = 0, 1, 2, \ldots n-1)$, $f_i(x,y)$ being defined only in the interval (x_i, x_{i+1}) where it is continuous, bounded, and satisfies a Lipschitz condition. Whatever be the subdivisions, all the quantities $|f_i|$ admit in their respective intervals of definition the same upper boundary M′, and the same coefficient K′ of the Lipschitz conditions

$$(4\cdot5) \quad |f_i(x,y)| < M' \quad |f_i(x,y') - f_i(x,y'')| < K' \cdot |y' - y''| \qquad (i = 0, 1, \ldots, n-1)$$

Then the polygonal line

$$(4\cdot5) \quad Y = y_0 + F(x_0,y_0)(x_1 - x_0) + F(x_1 y_1)(x_2 - x_1) + \ldots + F(x_{n-1},y_{n-1})(A - x_{n-1})$$

where

$$(4\cdot6) \qquad F(x_i,y_i) = f(x_i,y_i) + (x_{i+1} - x_i) f_i(u,y_i) \qquad x_i \leqslant u \leqslant x_{i+1}$$

will also tend to the same limiting curve C. The demonstration is exactly the same as in the first case referred to.

* This condition is satisfied if X admits continuous bounded first partial derivatives in (3·2) while these derivatives themselves satisfy a Lipschitz condition. (Cf. G. D. Birkhoff, *Dynamical Systems*, Amer. Math. Soc. Coll. Publications, vol. IX, chap. I).

† Cf. Goursat—Hedrick, *Mathematical Analysis*, II—2, p. 68-74.

5. *Motion of a particle of varying mass: First case: $X=0$.*

Let $m(t)$ be a monotone function of t satisfying the conditions stated in § 1 for a varying mass. We then write m_i for $m(t_i)$.

We have

(5·1) $$m_i \frac{d^2x}{dt^2} = 0$$

within the interval (t_i, t_{i+1}), $(i=0, 1, 2, \ldots n-1)$.

Hence

(5·2) $$x = a_i\ t + b\,;$$

in the same interval, and

$$a_i = \left(\frac{dx}{dt}\right)_{t=t_i+0}$$

Applying the theorem on impulse we get

(5·3) $$m_i\, a_i = m_{i-1}\, a_{i-1} \qquad (i=1, 2, \ldots m)$$

and therefore

$$a_i - a_{i-1} = -a_{i-1}\frac{m_i - m_{i-1}}{m_i}$$

or

(5·4) $$\Delta a_i = -a_i \frac{\Delta m_i}{m_i + \Delta m_i} \qquad (i=0, 1, \ldots n-1)$$

When the number of the subdivisions increases so that the greatest subinterval tends to zero, the polygonal line defined by (5·4) tends to a curve C whose differential equation is

(5·5) $$da = -a\frac{dm}{m}$$

for we are fulfilling the conditions of § 4. Hence $a = k/m$, k being a constant and the polygonal line tends to the curve of equation

(5·6) $$x = \int_{t_0}^{t} a dt = \int_{t_0}^{t} \frac{kdt}{m}$$

Now we have along that curve

(5·7) $$\frac{dx}{dt} = \frac{k}{m}, \qquad \frac{d}{dt}\left(m\frac{dx}{dt}\right) = 0$$

This is the law of motion along a linear path of a particle of varying mass.

6. *Second case: the force acting on the particle is propotional to its mass.*

In each interval (t_i, t_{i+1}) we have

(6·1) $$m_i \frac{d^2x}{dt^2} = m_i\ X$$

We introduce the function ξ defined by

$$\frac{d^2\xi}{dt^2}=X, \qquad \xi(t_0)=x(t_0), \qquad \dot{\xi}(t_0)=\dot{x}(t_0).$$

Thus we have in each interval (t_i, t_{i+1}).

(6·2) $$x-\xi=a_i t+b_i$$

with $$a_i=(\dot{x})_{t=t_i+0}-\dot{\xi}(t_i)$$

Let us put

(6·3) $$(\dot{x})_{t=t_i+0}=\dot{x}_{ir}, \qquad (\dot{x})_{t=t_i-0}=\dot{x}_{il} \qquad \dot{\xi}(t_i)=\dot{\xi}_i.$$

Then the law of impulse gives us at the point of abscissa t_i

$$m_i \dot{x}_{ir}=m_{i-1}\dot{x}_{il}.$$

Then a_i being a constant within (t_i, t_{i+1}) it follows that

$$\dot{x}_{il}-\dot{\xi}_i=\dot{x}_{i-1\cdot r}-\dot{\xi}_{i-1}$$

or

$$\dot{x}_{il}=\dot{x}_{i-1.r}+\dot{\xi}_i-\dot{\xi}_{i-1}.$$

Hence

$$\dot{x}_{ir}-\dot{x}_{i-1.r}=\dot{x}_{i-1.r}\frac{m_{i-1}-m_i}{m_i}+\frac{m_{i-1}}{m_i}(\dot{\xi}_i-\dot{\xi}_{i-1})$$

or

(6·4) $$\begin{cases}\Delta\dot{x}_{ir}=-\dot{x}_{ir}\dfrac{\Delta m_i}{m_i+\Delta m_i}+\dfrac{m_i}{m_i+\Delta m_i}\Delta\dot{\xi}_i\\ \dot{x}_{ir}=a_i+\dot{\xi}_i \end{cases} \qquad (i=0,1,2,\ldots,n-1)$$

Therefore we can apply the principles of § 4. Hence (6·4) defines a polygonal line which, when we pass to the limit, defines a function a of m. Then the curve of equation

$$y=\int_{t_0}^{t} a\,dt$$

will be the limit of the polygonal (6·2) and it is easy to see that along that curve we have

$$\frac{d}{dt}\left(m\frac{dx}{dt}\right)=m\frac{d^2\xi}{dt^2}=mX.$$

7. *Third case: the force is not proportional to the mass.*

We have now in each interval (t_i, t_{i+1})

(7·1) $$m_i\frac{d^2x}{dt^2}=X \qquad (i=o, 1, \cdots n-1)$$

and we associate with this equation as above the equation

$$\frac{d^2\xi}{dt^2}=X.$$

Hence we have in each interval

(7·2) $$m_i x - \xi = a_i t + b$$

with

(7·3) $$a_i = m_i \dot{x}_{ir} - \dot{\xi}_i$$

the notations being the same as in § 6.

We have also

$$m_i \dot{x}_{ir} = m_{i-1} \dot{x}_{il} \qquad (i = 1, 2, \cdots, n-1)$$

and a_i being a constant in the interval (t_i, t_{i+1}), it follows that

$$m_{i-1} \dot{x}_{i-1.r} - \dot{\xi}_{i-1} = m_{i-1} \dot{x}_{il} - \dot{\xi}_i.$$

Therefore :

(7·4) $$m_{i-1} \dot{x}_{i-1.r} - \dot{\xi}_{i-1} = m_i \dot{x}_{ir} - \dot{\xi}_i = k \qquad (i = 1, 2, \cdots, n)$$

k being a constant.

Now the equations

$$y_i = x - \frac{\xi}{m_i} = \frac{a_i}{m_i} t + \frac{b_i}{m_i} \qquad t_i \leqslant t \leqslant t_{i+1} \qquad (i = 0, 1, \cdots, n-1)$$

do no longer represent a polygonal line but they represent a discontinuous curve composed of straight lines of gradient $\frac{k}{m_i}$ in the intervals (t_i, t_{i+1}). We join the origins of these segments by straight lines. We thus obtain a polygonal line of gradient.

(7·5) $$p_i = \frac{k}{m_i} - \xi_{i+1}\left(\frac{1}{m_{i+1}} - \frac{1}{m_i}\right)\Big/(t_{i+1} - t_i) \qquad (i = 0, 1, \cdots, n-1)$$

in the interval (t_i, t_{i+1}), or of equation

(7·6) $$Y = y_0 + p_0(t_1 - t_0) + p_1(t_2 - t_1) + \cdots$$

We can easily verify that this equation satisfies the conditions of § 4. Therefore, passing to the limit, we obtain a curve C. Instead of joining the origins of the above mentioned segments we can as well join their extremities, and we then obtain another polygonal line tending manifestly to the same limiting curve C. Therefore the discontinuous curve composed of the segments themselves tends also to the limiting curve C (*).

(*) It is important to point out that, while the points of the discontinuous curves of equation $y_i = x - \xi/m_i$ in (t_i, t_{i+1}) $(i = 0, 1, 2, \cdots n-1)$ admit C as their limiting set, the set of the tangents to the discontinuous curves does not admit the set of the tangents of C as a limiting set (neighbourhood of order zero). But it is enough to prove that the paths generated by the successive subdivisions of $(t_0\, T)$ tend to C, because C considered as the limiting path defines a limiting motion and the differential equation, which C verifies, express the differential law of that limiting motion, which conventionally defines the motion of a particle of varying mass.

We have along that curve

$$\frac{dy}{dt} = \frac{k}{m} + \frac{\xi}{m^2}\frac{dm}{dt} \tag{7·7}$$

But it is evident that

$$y = x - \xi/m \tag{7·8}$$

Hence

$$\frac{dx}{dt} - \frac{1}{m}\frac{d\xi}{dt} + \frac{\xi}{m^2}\frac{dm}{dt} = \frac{k}{m} + \frac{\xi}{m^2}\frac{dm}{dt} \tag{7·9}$$

or

$$m\frac{dx}{dt} - \frac{d\xi}{dt} = k$$

and finally we can write

$$\frac{d}{dt}\left(m\frac{dx}{dt}\right) = \frac{d^2\xi}{dt^2} = X. \tag{7·10}$$

GLEANINGS

40. " As Lord Sri Narayana, who relieves the sufferings of birth and death, who is the sole primary cause of the creation of the universe, pervades this universe through His own manifestations as worlds, paradises, mountains, rivers, gods, men, demons, etc, so does the Rule of Three pervade the whole of the science of calculation..... Whatever is computed whether in algebra or in this (arithmetic) by means of multiplication and division may be comprehended by the sagacious learned as the Rule of Three. What has been composed by the sages through the multifarious methods and operations such as miscellaneous rules, etc. teaching its easy variations, is simply with the object of increasing the comprehension of the duller intellects like ourselves."

BHASKARA II, on the Rule of Three in the *Lilavati*.

41. " In this quantity consisting of that which has cipher for its divisor there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth."

BHASKARA II, *Bijaganita*.

OBSERVATIONS ON THE PRECEDING PAPER

BY

A. NARASINGA RAO, *Annamalainagar*

1. Father Racine's valuable observations seek to fill a gap in our text-books on Dynamics and will be read with interest by students and teachers of the subject. He shows that Newton's Fundamental Law (1·1) (*vide* page 58) not only yields (1·2) as a particular case when m is constant, but may itself be obtained from the latter by a passage to the limit.

It is however obvious that (1·2) *by itself* cannot yield (1·1) and that there exist alternatives to it from which also (1·2) may be deduced as a special case when $dm/dt = 0$. It is worth noting that the additional principle which has to supplement (1·2) in order to lead to (1·1) is the principle of conservation of momentum (2·1), and that (1·1) is the common parent of both. This will be clearer if we re-write (1·1)—confining ourselves to one dimension—in the form

$$\dot{x} = v\,; \qquad mdv + vdm = Xdt \qquad \ldots \quad (1{\cdot}1)$$

whence we get both

$$mdv - Xdt = 0 \qquad \ldots \quad (1{\cdot}2)$$

and

$$mdv + vdm = 0 \qquad \ldots \quad (2{\cdot}1)$$

as special cases corresponding to $dm = 0$ and $dt = 0$. Accordingly the author breaks up the motion into spells in which alternately the mass is constant ($t_i \rightarrow t_{i+1}$), and time is arrested $\dot{x}_i \rightarrow \dot{x}_{i+1, i}$.

We may thus write

$$(1{\cdot}1) \rightleftarrows (1{\cdot}2) + (2{\cdot}1) \qquad \ldots \quad (A)$$

A third possible deduction from (1·1) is

$$vdm = Xdt\,; \qquad \dot{v} = 0 \qquad \ldots \quad (1{\cdot}3)$$

which, it is reasonable to suppose, may be used as the principle for extending (1·2) in place of (2·1). This would mean alternate spells of constant mass (1·2) and constant velocity (1·3), the mass varying with the time during the second of these.

For example when the force X is proportional to the mass, say $X = km$ let us subdivide each subinterval (t_i, t_{i+1}) into two (t_i, t'_i) and

(t'_i, t_{i+1}) during the first of which $m = m_i$ remains constant and (1·2) holds, so that

$$m\dot{v} = X = km, \qquad t_i \leqslant t < t'_i$$

Hence

$$v'_i = v_i + k(t'_i - t_i)\,; \qquad v(t_i) = v_i\,; \quad v(t'_i) = v'_i \qquad \ldots \quad (1\cdot4)$$

During the next interval (t'_i, t_{i+1}), we have v constant so that

$$v_{i+1} = v'_i = v_i + k(t'_i - t_i) \qquad \ldots \quad (1\cdot5)$$

while by (1·3) $\quad v_{i+1}\dfrac{dm}{dt} = X = km, \qquad (t'_i \leqslant t < t_{i+1})$

$$\therefore \qquad m_{i+1} = m_i \exp\{k(t_{i+1} - t'_i)/v_{i+1}\} \qquad \ldots \quad (1\cdot6)$$

From (1·5) and (1·6) we have

$$m_{i+1}\,v_{i+1} = m_i\,v_i + m_i\,k\,\{(t'_i - t_i) + (t_{i+1} - t'_i)\,v_i\,/\,v_{i+1}\} + O(t_{i+1} - t_i)^2$$
$$= m_i\,v_i + m_i\,k\,(t_{i+1} - t_i) + O(t_{i+1} - t_i)^2$$

and this leads to the differential equation

$$\frac{d}{dt}(mv) = mk = X$$

All that has been said so far is intended to throw more light on the passage from (1·2) to (1·1) and not with a view to question the appropriateness of (2·1) as the link by which the author has effected such a transition. Its superiority over (1·3) rests both on its closer relation to physical phenomena (impulsive motion) and on its not involving the force X which varies from one problem to another.

2. Since (1·2) is a special case (m constant) of (1·1) while (2·1) provides an extension principle to the case of m variable, we may rewrite (A) in general terms

General case $\rightleftarrows$ Special case + Principle of extension. ... (B)

Usually the "principle of extension" gives differential or small-scale (im kleinen) properties and (B) represents an important mathematical process, that of passing from these to large scale or (im grössen) properties. It would be of interest to recognise (B) in its various special forms in different fields of mathematical thought.

In Analysis which is a basic mathematical discipline, the extension-principle for the functional relation is furnished by the derivate. Accordingly, we find (B) embodied in

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(x)\,dx$$

a result even more fundamental than Taylor's Theorem in that it does not assume the existence of higher derivates, while Taylor's theorem may itself be deduced by its repeated application. For functions of several variables it takes the form

$$f(x, y) = \int_{x_0, y_0}^{x, y} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

but there is here an "integrability criterion" by way of restriction on the principle of extension embodied in f_x and f_y namely $f_{xy} = f_{yx}$ to ensure that the value does not depend on the path of integration.

In geometry we have a close analogy to the dynamical discussion, in passing from the formula

$$s = \int (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} dt \qquad \dots \quad (2{\cdot}1)$$

to the special case when the curve is a straight line, namely

$$l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \qquad \dots \quad (2{\cdot}2)$$

For the reverse process of obtaining (2·1) from (2·2) by taking a curve as the limit of a broken line it is sufficient to assume

(i) that arc lengths are additive and

(ii) that the metrical structure is of the Finsler* type.

In the geometry of curves we create the concepts of "tangent" and "osculating plane" as aids to extension and the "principle of extension" is embodied in the relations between the metrical invariants of neighbouring points, tangents and osculating planes *i.e.* in the two curvatures. The result that a curve is uniquely determined by its initial elements and its two curvatures as functions of the arc corresponds to (B).

In differential geometry, the principle of extension of vectors is the affine connection and the "criterion of integrability" is the condition for teleparallelism, that is, the vanishing of the curvature tensor.

* For, (2·2) applied to all possible infinitesimally near points will detect a dependence of the metrical tensor on first derivates, but not on second derivates since these vanish for all straight lines.

ON THE NATURE OF CONTACT BETWEEN $S=0$ and $S-\lambda T=0$.

BY

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§ 1. In a note on this topic published in the "*Math. Student*" Vol. III, No. 4, Dr. S. S. Pillai establishes that the contact at a point (x_1, y_1) between the curves $S_1 \equiv F(x, y)=0$ and

$S_2 \equiv S_1 - \lambda\,(x-x_1)\partial F/\partial x_1 - \lambda(y-y_1)\,\partial F/\partial y_1 = 0$ where $F(x_1, y_1)=0$, is internal or external according as $\lambda < 1$ or $\lambda > 1$. His method of proof is by using the expression for curvature, and assumes the co-ordinates to be rectangular. The present note gives a direct method, essentially geometrical, which holds good for any set of Cartesian axes, and which can be extended to three-dimensional geometry.

Let

$$S \equiv a(x-x_1)+b(y-y_1)+A(x-x_1)^2+2H(x-x_1)(y-y_1)+B(y-y_1)^2+ \ldots\ldots\ldots\ldots$$

Then

$$T \equiv (x-x_1)\frac{\partial S}{\partial x_1}+(y-y_1)\frac{\partial S}{\partial y_1}=a(x-x_1)+b(y-y_1).$$

Applying the general linear transformation

$$x-x_1=l_1X+m_1Y; \qquad y-y_1=l_2X+m_2Y,$$

the expressions S and T change into

$$S' \equiv a'X+b'Y+A'X^2+2H'XY+B'Y^2+\ldots\ldots\ldots; \quad T' \equiv a'X+b'Y$$

Hence $S-\lambda T$ changes into $S'-\lambda T'$. In other words, the value of λ remains unaltered when the equations of the curves $S=0$, $S-\lambda T=0$ are subjected to any linear transformation accompanied by a change of origin. This means at once that the results obtained remain true for any set of Cartesian axes.

We shall now take the equations of two curves touching at the origin, in their simplest forms

$$S_1 \equiv y+ax^2+2hxy+by^2+\alpha x^3+\ldots\ldots=0 \qquad \ldots \quad (1)$$

$$S_2 \equiv \lambda y+ax^2+2hxy+by^2+\alpha x^3+\ldots\ldots=0 \qquad \ldots \quad (2)$$

Hence $S_2=S_1-(1-\lambda)T_1$. The contact is internal or external according as the curves are on the same side or on opposite sides of

the tangent*. In the neighbourhood of the origin, the curves can be approximately considered as conics. The line $y=\varepsilon$ where ε is sufficiently small meets the curves where

$$x^2\{a+O(\varepsilon)\}+2x\{h\varepsilon+O(\varepsilon)^2\}+\varepsilon+O(\varepsilon^2)=0 \qquad \ldots \quad (3)$$

$$x^2\{a+O(\varepsilon)\}+2x\{h\varepsilon+O(\varepsilon^2)\}+\lambda\varepsilon+O(\varepsilon^2)=0 \qquad \ldots \quad (4)$$

respectively. Suppose now that ε is chosen so that (3) gives real roots. This requires $a\varepsilon$ to be negative.† It then follows that the roots of (4) are real or imaginary according as $\lambda>0$ or $\lambda<0$. Dr. Pillai's result follows, noting that the λ here is equal to his $1-\lambda$.

§ 2. The corresponding problem in three-dimensional geometry will present in one case a question of interpretation. In the neighbourhood of a real ordinary point on a surface, the expansion of the surface can be written

$$2z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+ax^3+\ldots\ldots$$

where ρ_1 and ρ_2 are the principal radii of curvature. If the origin is an elliptic (synclastic) point, the indicatrix $z=k$; $\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}=2k$ is an ellipse which is real or imaginary according as $k>0$ or $k<0$. But if the origin is a hyperbolic (anticlastic) point, the indicatrix is a hyperbola which is real for either sign of k, but the disposition of the hyperbola is different for the two signs of k, the transverse axis of the hyperbola for one sign of k being parallel to the conjugate axis for the other sign.

Therefore, if we consider the two surfaces

$$2z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\ldots\ldots \qquad \ldots \quad (5)$$

$$2\lambda z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\ldots\ldots \qquad \ldots \quad (6)$$

it is evident that when the origin is an elliptic point, the two surfaces are on the same side of the tangent plane if and only if $\lambda>0$.

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* In ordinary cases, this is the same as saying that the centres of curvature are on the same side or on opposite sides of the tangent, as will be clear from a figure. A formal proof is easily given. But in special cases, the curvature definition will not be so convenient as the definition adopted here. Consider for example the curves $y=x^2$ and $y=\pm x^4$. The origin is a point of undulation for the latter, so that the centre of curvature is at infinity. A given line has only one point at infinity. The curvature definition has therefore to be suitably modified to suit such cases.

† We take $a\neq 0$. Otherwise the origin would be a point of inflexion on either curve.

But suppose that the origin is a hyperbolic point. Then, if $\lambda > 0$, the dispositions of the indicatrices

$$z = k\ ;\ 2k = \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} \quad \text{and} \quad z = k\ ;\ \ 2\lambda k = \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2}$$

will be similar: while if $\lambda < 0$, the disposition of the hyperbola $z = k\ ;\ \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} = 2\lambda k$ will be similar to the section of the first surface by $z = -k$ i.e. by a plane *on the other side* of the tangent plane. With this meaning, we can for this case say that the two surfaces are "on opposite sides" at the point, or "touch each other externally". We have now the following result:

The two surfaces $S \equiv F(x, y, z) = 0$ *and* $S - \lambda T = 0$, *where*

$$T \equiv (x - x_1)\frac{\partial F}{\partial x_1} + (y - y_1)\frac{\partial F}{\partial y_1} + (z - z_1)\frac{\partial F}{\partial z_1}, \quad \textit{and} \quad F(x_1, y_1, z_1) = 0$$

touch each other internally or externally according as $\lambda < 1$ *or* $\lambda > 1$.

We shall work out the proof taking the finite equations of the surfaces; it is not sufficient to infer this from the discussion of the standard forms (5) and (6) in terms of infinite series. But we can take the equations of the surfaces in the simplest forms, since as in § 1, a linear transformation accompanied by a change of origin will not affect the value of λ. Hence, referring the axes to the principal directions and the normal at the point of contact, the equations of the surfaces can be taken as

$$z + (ax^2 + by^2 + cz^2 + 2fyz + 2gzx) + \text{higher terms} = 0 \qquad \ldots \qquad (7)$$

$$\mu z + (ax^2 + \cdots\cdots\cdots) + \cdots\cdots = 0 \qquad \ldots \qquad (8)$$

The section of (7) by the plane $z = \varepsilon$ where ε is sufficiently small is the conic

$$Ax^2 + 2Hxy + By^2 + 2\varepsilon(Fy + Gx) + (\varepsilon + C\varepsilon^2) = 0,$$

where the capitals differ from the corresponding small letters by amounts of the order of ε, the value of h being zero. Transferring this conic to its centre, its equation can be written

$$AX^2 + 2HXY + BY^2 + \varepsilon + O(\varepsilon^2) = 0 \qquad \ldots \qquad (9)$$

The section of (8) similarly gives

$$AX^2 + 2HXY + BY^2 + \mu\varepsilon + O(\varepsilon^2) = 0. \qquad \ldots \qquad (10)$$

It is easy to see that if (9) is a real ellipse, (10) is an ellipse which is real or imaginary according as $\mu > 0$ or $\mu < 0$; while if (9) is a real hyperbola, the transverse axes of (9) and (10) lie along the same

line when $\mu>0$, but the transverse axis of one lies along the conjugate axis of the other if $\mu<0$. Since μ is equal to $1-\lambda$, the theorem follows.

Note: Prof. A. Narasinga Rao has suggested that the problem may be generalised by taking the polar conic operator

$\left[(x-x_1)\frac{\partial}{\partial x_1}+(y-y_1)\frac{\partial}{\partial y_1}+(z-z_1)\frac{\partial}{\partial z_1}\right]^2$ in addition to the tangent operator. We have thus the following result:

The curves $S\equiv F(x, y, z)=0$ and $S-\lambda T+\frac{1}{2}\mu C=0$, where

$$T\equiv(x-x_1)\frac{\partial S}{\partial x_1}+(y-y_1)\frac{\partial S}{\partial y_1}+(z-z_1)\frac{\partial S}{\partial z_1};\quad C=\left[\sum(x-x_1)\frac{\partial}{\partial x_1}\right]^2 S;$$

and $F(x_1, y_1, z_1)=0$, touch each other at (x_1, y_1, z_1) internally or externally according as $\frac{1-\lambda+(n-1)\mu}{1+\mu}>0$ or <0, the co-ordinates being cartesian, and $z=1$ being introduced to render the equations homogeneous.

A similar result holds for three dimensions.

ON THE CONTACT OF VARIETIES IN n-SPACE

BY

A. NARASINGA RAO, *Annamalainagar*

Introduction

The object of this paper is twofold:

(i) to suggest an extension to real varieties in Euclidean n-space of the concepts of "internal contact" and "external contact," and to prove the analogue of Dr. Pillai's result regarding the contact of $S=0$, $S-\lambda T=0$ (*Math. Student*, Vol. III, P. 156). The extension to three dimensions is discussed by Dr. Srinivasiengar in the paper which precedes;

(ii) to show that the concepts defined above belong to the domain of projective differential geometry, that is, that they are conserved by real projective transformations.

1. Contact of plane curves: Extension to n-space

If two circles touch, the distance between the centres is equal to the sum or difference of the radii. In the former case we say the

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contact is "external" and in the latter "internal". Two curves are said to have external or internal contact at O according as their circles of curvature at O have the one or the other. The equations of two such curves may be thrown simultaneously into the canonical forms

$$(1\cdot1) \qquad y = a_1x^2 + a_2xy + a_3y^2 + \ldots\ldots ;$$
$$y = b_1x^2 + b_2xy + b_3y^2 + \ldots\ldots$$

the omitted terms being of the third or higher degree. The contact is external or internal according as a_1 and b_1 have like or unlike signs. Both here and throughout this paper the axes are not necessarily rectangular.

The properties associated with such contacts are

(1·2) there exists an interval I ($|x| < \varepsilon$) such that the ordinates y_1 and y_2 of the two curves for every abscissa x in the interval are of the same sign if the contact is internal and of opposite signs if the contact is external;

(1·3) Any straight line $y = \varepsilon_1 x + \varepsilon_2$ infinitesimally close to the tangent and meeting it ultimately at a point other * than the point of contact ($\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, lim. $\varepsilon_2/\varepsilon_1 \neq 0$) cuts both the curves in a pair of real points or both in a pair of imaginary points, in the case of internal contact. For external contact, such a line meets one curve in two real points and the other in a pair of imaginary points. In particular we may take a line parallel to the tangent and close to it ($\varepsilon_1 = 0$). The intersections considered are only those whose abscissa lies in the interval I.

(1·4) When the two curves touch internally, the segment of the transversal in (1·3) cut out by one curve is contained wholly in the segment cut out by the other; in other words, the two point pairs cut by the two curves determine an involution whose double points are real.

Results (1·2), (1·3), (1·4) suggest analagous properties in n-space which may be used as definitions to characterise the contact of two varieties. It will be more instructive, however, to use definition (1·5) given below and to examine how far the properties (1·2), (1·3), (1·4) follow as consequences therefrom.

* It is easily seen that the property mentioned is not true if the ultimate intersection of the transversal with the X-axis is the point of contact (0, 0); for example take the line parallel to and mid-way between parallel tangents to the two curves.

DEFINITION :

(1·5) *Two varieties in* [n]* *which touch at O are said to have "external" or "internal" contact according as every plane (flat two space) section of them through O but not lying in the tangent prime*† *at O gives curves which have external or internal contact. If the contact is external for some sections and internal for others, the contact will be said to be "neutral."*

2. Second order neighbourhood : asymptotic forms

Any surface in [3] which touches the plane $z=0$ at the origin O has the equation

(2·1) $$z = ax^2 + 2hxy + by^2 + \quad \dots \quad \dots$$

where the omitted terms are those in z^2, zx, zy and those of the third and higher orders in x, y, z. It is clear from (2·1) that if x, y are infinitesimals of the first order, z is of order two and that its principal part consists of the terms written down. In fact, the Taylor expansion for z up to terms of the second order are just those written down in (2·1).

Similarly, the equation of a variety in [n] touching the prime $x_n=0$ at the origin, may be written.

(2·2) $$x_n = \sum a_{rs}x_rx_s + \sum a_{rst}x_rx_sx_t \dots \qquad (r,s,t=1,2\dots n)$$

whence $\partial x_n/\partial x_r = 0$, $\partial^2 x_n/\partial x_r\partial x_s = a_{rs}$ $(r, \ s=1, \ 2\dots n-1)$ at the origin. Hence the second order neighbourhood of the variety near O – the Taylor expansion up to terms of the second order – is given by

$$x_n = \sum a_r \, x_rx_s \qquad (r, \ s=1, \ 2 \ \cdots \quad n-1)$$

The quadratic form $\sum a_{rs}x_rx_s$ is bound to play an important part in all curvature properties and may be called the *Asymptotic form* ‡ of the variety at O, since when equated to zero it gives the asymptotic directions at O. It may be obtained from the terms of degree two in the second member of the canonical equation (2·2) by putting $x_n=0$.

3. Criteria for Internal and External Contact in [n]

We shall now prove the main theorem of the paper, namely,

(3.1) *Two varieties in* [n] *which touch at O have non-neutral contact when their asymptotic forms at O are either both definite, or if indefinite*

* [n] stands for Euclidean space of n-dimensions.

† A prime is a [$n-1$] in [n].

‡ The "asymptotic form" is a scalar multiple of the second fundamental quadratic differential form when $x_1x_2 \cdots x_{r-1}$ are taken to be parameters.

one is a multiple of the other. In the former case, the contact is internal or external according as the definite forms are of like or of unlike sign; in the latter according as the numerical multiplier is positive or negative. In every other case, the contact is neutral.

For, let the two varieties be given by (2·2) and a similar equation with a_{rs}, a_{rst} etc. replaced by b_{rs}, b_{rst} etc. Consider now their sections by the plane containing the lines $x_1/\lambda_1 = x_2/\lambda_2 = \ldots x_n/\lambda_n$ and $x_1/\mu_1 = x_2/\mu_2 = \ldots x_{n-1}/\mu_{n-1}$, $x_n = 0$ the latter being the section of the plane by $x_n = 0$.

We now introduce new co-ordinates (X) given by

(3.2) $x_r = \sum\limits_s c_{rs} X_s$; $x_n = \lambda_n X_n$ $\quad (r = 1, \ldots n-1)$; $(s = 1, \ldots n)$

If we take

(3·3) $c_{r1} = \mu_r$; $\quad c_{rn} = \lambda_r$

the two lines are carried over into the X_n and X_1 axes. The equations of the varieties change over into

(3·4) $X_n = \sum\limits_s A_{rs} X_r X_s + \ldots \ldots$; $\quad X_n = \sum\limits_s B_{rs} X_r X_s + \ldots$

where

(3·5) $A_{rs} = (\sum\limits_{pq} a_{pq} c_{pr} c_{qs})\lambda_n$; $\quad B_{rs} = (\sum\limits_{pq} b_{pq} c_{pr} c_{qs})/\lambda_n$

The sections by the $X_1 X_n$ plane are now given by

$$X_n = A_{11} X_1^2 + \ldots \; ; \quad X_n = B_{11} X_1^2 + \ldots \ldots$$

the omitted terms containing terms $X_1 X_n$, X_n^2 and higher degree terms. The contact of the two curves is internal or external according as

(3·6) $A_{11}/B_{11} = (\sum a_{pq} \mu_p \mu_q)/(\sum b_{pq} \mu_p \mu_q)$

{using (3·5) and (3·3)} is positive or negative. Hence in order that the contact between the sections may be always internal or always external the sign of the ratio A_{11}/B_{11} should be independent of the values μ_r. Thus two cases arise:

Case (*i*): Each of the two asymptotic forms $\sum a_{rs} \mu_r \mu_s$ and $\sum b_{rs} \mu_r \mu_s$ $(r, s = 1, 2 \ldots n-1)$ may be definite—of the same sign for all values of the variables. The contact is then internal or external according as they are of the same or opposite signs.

Case (*ii*): If the forms are not both definite, every set of values which make one form positive must make the other also positive (internal contact), or must make the other negative (external) in every

case. This implies that the two quadratic forms vanish together and hence that one is a numerical multiple of the other. The contact is internal or external according as the multiplier is positive or negative.

If neither of these holds, the sections touch sometimes internally and sometimes externally and the varieties have neutral contact.

4. Analogues with contact in plane

We now examine how far the properties (1·2), (1·3), (1·4) have analogues in $[n]$.

By what has been stated in connection with (2·2), there exists a region $|x_r| < \delta$ $(r=1, 2 \cdot\cdot n-1)$ such that x_n has the same sign as the form $\Sigma a_{rs} x_r x_s$ $(r, s=1 \cdot\cdot\cdot\cdot n-1)$. Hence the ratio of x_n for the varieties corresponding to any point in the region has the same sign as the ratio of the two asymptotic forms and is positive for internal and negative for external contact. Hence (1·2) *has an analogue in* [n]

The section of (2·2) by the parallel prime $x_n = \pm \varepsilon$ give the indicatrix quadrics.

(4·1) $\sum a_{rs}\ x_r\ x_s \mp \varepsilon\ x_o^2 = 0\ ;\ \sum b_{rs}\ x\ \ x_s \mp \varepsilon\ x_o^2 = 0$

where x_o is an additional variable introduced for homogeneity. By real linear transformations we may reduce any quadratic form to an algebraic sum of squares and in whatever way it is done, the number of positive terms p and the number of negative terms q are the same. Let us call (p, q) the characteristics of the form. It is now easy to verify that in every case of internal contact the sections (4·1) have the same characteristics while for external contact the section of the one by $x_n = +\varepsilon$ has the same characteristics as the section of the other by $x = -\varepsilon$. This is the analogue of (1·3) in $[n]$.

The analogue of (1·4) namely, that all values of $x_1 \ldots x_{n-1}$ which make one asymptote form vanish, give the same sign to the other—in other words that the two sections (4·1) have no real intersection is only true in case (ii) when, one form is a multiple of the other. When the forms are both definite and of the same sign, this implies a further condition that the difference of the two forms is also definite.

5. It is now easy to extend to n-space Dr. Pillai's result (Student, Vol. 3, P. 156) extended by Dr. Srinivasiengar to [3]. Let F be the variety (2·2). Then T is the result of applying the polar operator $x_1 \partial/\partial\xi_1 + \ldots x_n \partial/\partial\xi_n$ to F where $\xi_1 = \ldots \xi_n = 0$, so that $T \equiv x_n$. Thus $F - \lambda T = x_n(1-\lambda) - \sum a_{rs} x_r x_s - \sum a_{rst}\ x_r x_s x_t \ldots$. Hence the asymptotic

forms of $F=0$ and $F-\lambda T=0$ when expressed in the canonical form (2·2) differ only by a numerical multiplier $1-\lambda$. Hence the contact is internal or external according as $\lambda < $ or > 1.

6. Character of the contact unaltered by projective transformations

We shall now prove that internal and external contact are projective differential properties. Recalling to mind the definition (1·5) it is sufficient, since plane sections transform into plane sections to prove the theorem for plane curves. Even here we may limit ourselves to the sub-group which leaves the point of contact (0, 0) and the tangent thereat ($y=0$) invariant, since we may by a metrical transformation (rigid motion) bring them to their old positions and such a displacement does not affect curvature and therefore also the character of the contact. Now the transformation

$$x=(c_1 X+c_2 Y)/D\,; \qquad y=c_3 Y/D\,; \qquad D=l_1 X+l_2 Y+l_3$$

carries the two curves (1·1) into others which may be thrown into the forms

$$Y=A_1X^2+A_2XY+A_3Y^2+\ldots\ldots$$

$$Y=B_1X^2+B_2XY+B_3Y^2+\ldots\ldots$$

where $A_1=a_1c_1{}^2/c_3l_3$ $B_1=b_1c_1{}^2/c_3l_3$; so that $A_1/B_1=a_1/b_1$. It is thus seen that, not only the sign of a_1/b_1 which determines the nature of the contact, but also its value is a projective invariant. It is, in fact, the limiting cross ratio of the range $(a\ t\ b\ f)$ in which f is a fixed point not on the tangent, a, b, t are the intersections of a transversal through f with the 2 curves and the common tangent, when the point t tends to the point of contact.

7. Extension to inflexional contact of any order

Two curves which have r point contact with the tangent at O may be written in the form

$$y=ax^r+\ldots\ldots\ldots\,;\ y=bx^r+\ldots\ldots\ldots$$

where the omitted terms are of order r in x, y and of higher orders. We may call the contact "external" or "internal" according as a/b is positive or negative. As in the case of $r=2$, a/b is a projective invariant.*

When r is even, properties (1·2), (1·3), (1·4) still hold. If r is odd there is only one intersection infinitesimally close to O. In the case of internal contact the abscissae of the intersection of any line des-

* Lane: *Projective Differential Geometry of Curves and Surfaces*, p. 30.

cribed in (1·3) with the two curves are of the same sign, for external contact they are of opposite signs. (1·4) does not arise for r odd.

Two surfaces in (3) or varieties in $[n]$ may be said to have inflexional contact of order r if every plane section through O gives curves which have such contact. If in every case the sections have internal (external) contact, the varieties will be said to have internal (external) contact.

Taking the origin at O and $x_n=0$ as the tangent prime as before the equations of the varieties take the form

$$x_n = \sum a_{lm\ldots\ldots} x_l x_m \ldots\ldots \qquad (l, m\ldots = 1, 2\ldots\ldots n) \tag{7·1}$$

Omitting all terms on the right which contain x_n or which are of order $r+1$, we have an "asymptotic form" of order r in $x_1\ldots\ldots x_{n-1}$, and if the ratio of two such forms is always of the same sign, the contact is non-neutral and is internal or external according as it is positive or negative. A necessary and sufficient condition for this is that the forms be either both definite or that one be a multiple of the other.

The linear polar operator is projective and applied to (7·1) yields x_n. Hence two varieties F [given by (7·1)] and F − λT have asymptotic forms differing by the multiplier 1 − λ. Hence the contact is internal or external according as λ < or >1.

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42. In this geometric symbolism God the Absolute or Unknowable is represented by a point, or dot (parm), which is one of the Hindu sectarial marks. The symbol of God manifested in the cosmos is an equilateral triangle, the three sides of which may be taken to represent the Trimurti. When the triangle stands on its apex, it signifies expansion or evolution, and, like the Swastika, the ascending creative force—or life. It is also the symbol of water as the creative element and is adopted by the Vaishnavaites as the symbol of Narayana-Vishnu. The triangle reversed or standing on its base signifies involution, or contraction, and hence fire, as the destructive element. It is one of the symbols of Siva.

The two triangles intersecting form the mystic lotus, known as King Solomon's Seal, the Seat of Brahma, the Casket which contained the Jewel of Life. It was also the symbol of the cosmic element, Ether.

F. B. HAVELL, *The Ideals of Indian Art.*

NOTES AND DISCUSSIONS

On the Expression of A Biquadrate as the sum of Three Biquadrates.

In a note under the above title,* Mr. Kothandaraman, following Mr. Brojomohan Padhy's detection of an error in Werebrusow's attempted proof of Euler's conjecture that

$$x^4 + y^4 + z^4 = u^4$$

is impossible in integers x, y, z, u all different from zero, tried to complete the proof by starting from the correct form of Werebrusow's identity. However, the proof is incomplete, and Euler's conjecture is unproved.

In case the reader does not have access to what Werebrusow stated** I give a literal translation, making the necessary correction of the misprint pointed out by Mr. Padhy in Werebrusow's formula. With this correction the statement is correct, *up to the last* (and fatal) *sentence.*

"Having rendered the three numbers squares in Euler's formula

$$(a^2+b^2+c^2+d^2)^2 = (a^2+b^2-c^2-d^2)^2 + (2ac+2bd)^2 + (2ad-2bc)^2,$$

I have found that

$$(3+4t^2)^4 = (-1+4t^2)^4 + (2+4t)^4 + 16[2\,(2t-1)(2t^2+t+1)\,]^2.$$

The last term cannot be a biquadrate. The equation is therefore impossible."

Even if the last term cannot be a biquadrate, and it cannot, as proved by Mr. Kothandaraman, *nothing whatever is proved regarding Euler's conjecture.* In order that the argument *should* give a proof, *it would be necessary to prove that Werebrusow's identity furnishes ALL cases of Euler's formula in which 'the three numbers are squares'.* Werebrusow offers no such proof; hence his conclusion (from the corrected form of his identity) that 'the equation is therefore impossible' is illegitimate.

Euler's conjecture is still open.

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* *Mathematics Student*, vol. 3, No. 4 (1935), pp. 155-56.

** *L'Intermediare des mathématiciens*, vol. 21, 1914, p. 161.

Note on Convergence of a Certain Series

The usual tests of convergence of a power series $\sum a_r/x^r$ depend on the ratio a_r/a_{r-1} of two consecutive coefficients. If the sequence of coefficients be defined by a recurring relation of the form

$$(A + Br)a_r - (C + Dr)a_{r-1} + (E + Fr)a_{r-2} = 0 \qquad (A)$$

with arbitrary values of a_0 and a_1, the usual tests of convergence will fail. Of course, without loss of generality a_0 may be taken to be unity. In the present note a particular case of the problem $A = 0$, $a_1 = C + D$ has been solved. It has been found that the series is equal to $\left(1 - \frac{a}{x}\right)^{-\alpha}\left(1 - \frac{b}{x}\right)^{-\beta}$ where a, b; α, β are constants depending on B, C, D, E, F.

If $A = 0$ we can take $B = 1$. Let a, b be the roots of the quadratic equation

$$x^2 - Dx + F = 0 \qquad \ldots \quad (1)$$

and let α, β be chosen so that

$$a(\alpha + r - 1) + b(\beta + r - 1) = C + Dr \qquad \ldots \quad (2)$$

and

$$ab(\alpha + \beta + r - 2) = E + Fr \qquad \ldots \quad (3)$$

Now because a and b are roots of the equation (1), (2) and (3) become

$$a\alpha + b\beta = C + D \qquad \ldots \quad (4)$$

and

$$\alpha + \beta = (E + 2F)/F \qquad \ldots \quad (5)$$

If a and b are real and different, so are α and β. If $a = b$, $\alpha + \beta$ may be uniquely determined and it is $\alpha + \beta$ that enters into the equations.

If $\quad |x/a| > 1$ and $|x/b| > 1$,

$$\left(1 - \frac{a}{x}\right)^{-\alpha} = 1 + \alpha \cdot \frac{a}{x} + \;\ldots\;\ldots\; + \frac{\alpha(\alpha+1)\cdots(\alpha+s-1)}{s\,!}\left(\frac{a}{x}\right)^s + \ldots\;\ldots$$

and

$$\left(1 - \frac{b}{x}\right)^{-\beta} = 1 + \beta \cdot \frac{b}{x} + \;\ldots\;\ldots\; + \frac{\beta(\beta+1)\cdots(\beta+t-1)}{t\,!}\left(\frac{b}{x}\right)^t + \cdots$$

Multiplying the two series

$$\left(1 - \frac{a}{x}\right)^{-\alpha} \cdot \left(1 - \frac{b}{x}\right)^{-\beta} = 1 + \frac{k_1}{x} + \frac{k_2}{x^2} + \cdots + \frac{k_r}{x^r} + \cdots$$

where $\quad k_r = \dfrac{\alpha(\alpha+1)\cdots(\alpha+r-1)}{r\,!}\,a^r + \dfrac{\beta(\beta+1)\cdots(\beta+r-1)}{r\,!}\,b^r$

$$+\sum_{s=1}^{s=r-1} \frac{\alpha(\alpha+1)\cdots(\alpha+r-s-1)}{(r-s)!} \frac{\beta(\beta+1)\cdots(\beta+s-1)}{s!} a^{r-s} b^s.$$

It will be shown that the sequence k_r so defined satisfies the same recurring relation.

$$rk_r = \frac{\alpha(\alpha+1)\cdots(\alpha+r-1)}{(r-1)!} a^r + \frac{\beta(\beta+1)\cdots(\beta+r-1)}{(r-1)!} b^r$$

$$+r \sum_{1}^{r-1} \frac{\alpha(\alpha+1)\cdots(\alpha+r-s-1)}{(r-s)!} \frac{\beta(\beta+1)\cdots(\beta+s-1)}{s!} a^{r-s}\cdot b^s.$$

Subtracting $\{a(\alpha+r-1)+b(\beta+r-1)\}k_{r-1}$ from this we get as the coefficient of $a^{r-s} b^s$

$$r \frac{\alpha(\alpha+1)\cdots(\alpha+r-s-1)}{(r-s)!} \cdot \frac{\beta(\beta+1)\cdots(\beta+s-1)}{s!}$$

$$-\frac{\alpha(\alpha+1)\cdots(\alpha+r+s-2)}{(r-s-1)} \times \frac{\beta(\beta+1)\cdots(\beta+s-1)}{s!}(\alpha+r-1)$$

$$-\frac{\alpha(\alpha+1)\cdots(\alpha+r-s-1)}{(r-s)!} \cdot \frac{\beta(\beta+1)\cdots(\beta+s-2)}{(s-1)!} \times (\beta+r-1).$$

The sum of the first two terms is equal to

$$\frac{\alpha(\alpha+1)\cdots(\alpha+r+s-2)}{(r-s)!} \cdot \frac{\beta(\beta+1)\cdots(\beta+s-1)}{(s-1)!}(\alpha-1).$$

Hence the coefficient of $a^{r-s} b^s$

$$= \frac{\alpha\cdots(\alpha+r-s-2)}{(r-s)!} \cdot \frac{\beta\cdots(\beta+s-2)}{(s-1)!}\{(\alpha-1)(\beta+s-1) - (\beta+r-1)(\alpha+r-s-1)\}$$

$$= -(\alpha+\beta+r-2) \frac{\alpha(\alpha+1)\cdots(\alpha+r-s-2)}{(r-s-1)!} \cdot \frac{\beta(\beta+1)\cdots(\beta+s-2)}{(s-1)!}$$

$= -(\alpha+\beta+r-2) \times$ the coefficient of $a^{r-s-1} b^{s-1}$ in k_{r-2}.

Hence the sequence k_r satisfies the relation (A) in its particular form. Because both the sequences are uniquely defined, $a_r = k_r$. Hence the series $\sum a_r/x^r$ is convergent and converges to

$$\left(1-\frac{a}{x}\right)^{-\alpha}\left(1-\frac{b}{x}\right)^{-\beta}.$$

A solution of the general problem set forth in the beginning should be of interest.

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V. RANGACHARIAR.

REMARKS BY V. GANAPATI IYER

A general method of summing the series

$$y = a_0 + a_1 z + a_2 z^2 + \cdot \quad \cdot \quad \cdot \qquad \ldots \quad (1')$$

where $z = 1/x$ and $a_n(A + Bn) - a_{n-1}(C + Dn) + a_{n-2}(E + Fn) = 0$ is indicated below.

If (1′) converges at all, it will do so in a circle $|z| < R$. Using the recurrence relation it can be verified that y is a solution of the differential equation

$$z(Fz^2 - Dz + B)\frac{dy}{dz} + y[(E + 2F)z^2 - (C + D)z + A]$$
$$= a_0 A + [a_1(A + B) - (C + D)a_0]z. \qquad \ldots \quad (2')$$

Now, (2′) is a linear differential equation of the first order whose coefficients are rational functions in z. In any given case, we can solve (2′) and determine that solution which is regular in the neighbourhood of $z = 0$ which takes the value $y = a_0$ for $z = 0$. For the case dealt with in the note, ($A = 0$, $B = a_0 = 1$, $C + D = a_1$), we have to solve

$$(Fz^2 - Dz + 1)\frac{dy}{dz} + y[(E + 2F)z - a_1] = 0. \qquad \ldots \quad (3')$$

The solution required is easily found to be $y = \left(1 - \frac{z}{\lambda}\right)^{\lambda'}\left(1 - \frac{z}{\mu}\right)^{\mu'}$ where λ, μ are the roots of $Ft^2 - Dt + 1 = 0$ and λ', μ' are numbers depending on C, D, E, F. But some caution is necessary in certain cases. It can happen that for arbitrary a_0 and a_1, (2) may possess no solution regular at $z = 0$. In such cases it means that (1′) cannot converge for $|z| > 0$. Even here it may be possible to choose a_0 and a_1 so as to get a regular solution (1′). As an illustration, consider the case where $A = 1$, $B = C = 0$, $D = -E = F = 1$. Here we have

$$z^2(z - 1)\frac{dy}{dz} + y(z^2 - z + 1) = a_0 + (a_1 - a_0)z \qquad \ldots \quad (4')$$

The solution of (4′) is

$$(z - 1)e^{1/z}y = k + \int [a_0 + (a_1 - a_0)z]e^{1/z}dz/z^2.$$

This can be regular only if $a_1 = a_0$ and even then only for $k = 0$. If all these be true, $y = -a_0/(z - 1)$ as can be verified directly. If $a_1 - a_0 \neq 0$, the series (1′) can converge only for $z = 0$ when a_n, $n \leqslant z$, is determined by the given recurrence relations.

A Fallacy connected with Newton's Theorem

In Askwith's *Pure Geometry*, Newton's Theorem is proved (page 117) by applying Carnot's Theorem to a triangle one of whose vertices is at infinity, the ratio of the two infinite sides being taken to be unity. Again later, Newton's Theorem is applied to a parabola (p. 136) and to a hyperbola (p. 177) in which cases also one of the intersections with the conic is at infinity. Perhaps these may be justified by a limiting process, but the method is not without danger when applied without due precautions. I give below an example of such reasoning leading to an obviously wrong result:

Newton's Theorem states that *if O be a variable point in the plane of a conic and PQ, RS be chords in fixed directions through O, then $OP \cdot OQ / OR \cdot OS$ is constant.*

Let S be the point at infinity on an asymptote of a hyperbola and RS, R′S two lines parallel to the asymptote and on the same side of it cutting the curve in R and R′. Then RR′ lie on the same branch of the hyperbola. Let a line O′ O parallel to RR′ cut the other branch at PQ and meet the lines RS, R′S in O and O′. Then Newton's theorem gives

$$\frac{OP \cdot OQ}{OR} = \frac{O'P \cdot O'Q}{O'R'} \quad \frac{OS}{O'S} = \frac{O'P \cdot O'Q}{O'R'}$$

since $OS/O'S = 1$. But this is absurd since the denominators OR, O′R′ are equal while one numerator is greater than the other, OO′ being both on the same side of PQ.

It is required to locate the error and to make clear the precautions in applying Newton's Theorem when one or both of the directions coincide with those of the asymptotes.

St. Joseph's College, *Trichy.* S. SURYANARAYANA IYER.

A Proof of Newton's Theorem

In view of the remarks made above regarding possible errors in using ratios of infinite line segments, the following proof of Newton's Theorem may be of interest.

Let ORS and OPQ be two chords of an ellipse and let O′ be a point in ORS and let O′ P′ Q′ be the chord parallel to OPQ.

By applying Pascal's Theorem to the hexagon QPSRQ′P′ we get that X, Y, Z (which are respectively the intersections of P′ Q, and

SR; PS and P′ Q′; RQ′ and PQ), are collinear. Hence by similar triangles, (with due regard to signs)

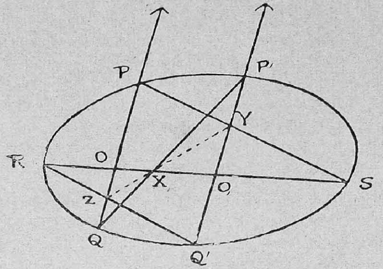

Fig. 1.

$$\frac{OX}{XO'} = \frac{OQ}{P'O'}; \quad \frac{O'Y}{OP} = \frac{O'S}{OS}; \quad \frac{Q'O'}{ZO} = \frac{RO'}{RO}; \quad \text{Also } \frac{OZ}{YO'} = \frac{OX}{XO'}$$

Multiplying these four equations we get that

$$\frac{O'P' \times O'Q'}{O'R \times O'S} = \frac{OP \times OQ}{OR \times OS}$$

i.e. we have proved Newton's Theorem when one chord is fixed and only one chord moves, parallel to itself.

Now if O_1 is a point in P′ Q′ and O_1 R′ S′ is a chord through O, parallel to ORS, by the previous part, we get similarly

$$\frac{O_1P' \times O_1Q'}{O_1R' \times O_1S'} = \frac{O'P' \times O'Q'}{O'R \times O'S}.$$

Hence
$$\frac{O_1P' \times O_1Q'}{O_1R' \times O_1S'} = \frac{OP \times OQ}{OR \times OS}.$$

None of the points P, Q, R, S should be at infinity.

2. *Extension of Newton's Theorem, to the case when some of the points* P, Q, R, S, *are at infinity.*

Case I. Parabola.

If one of the chords is parallel to the axis of the parabola then the theorem takes the form, $\frac{OP \times OQ}{O\,R} = \frac{O_1P_1 \times O_1Q_1}{O_1R_1}$ where ORS is the chord parallel to the axis and S is the point at infinity on the chord.

First, take O' in R S and let $O'P_1Q_1$ be parallel to OPQ. As before applying Pascal's theorem to the hexagon, $RSPQP_1Q_1$, we get that X, Y, Z are collinear. As before, by similar triangles,

$$\frac{OX}{XO'}=\frac{OQ}{P_1O'}; \quad O'Y=OP; \quad \frac{Q_1O'}{ZO}=\frac{RO'}{RO}; \quad \frac{OZ}{YO'}=\frac{OX}{XO'}$$

Multiplying these, we get the result $\frac{OP \times OQ}{OR}=\frac{O'P_1 \times O'Q_1}{O'R}$.

Secondly, let O_1 be a point in P_1Q_1 and let O_1R_1S be the chord parallel to $O'RS$.

Now applying Pascal's Theorem to the hexagon $RQ_1P_1R_1SS$ and remembering that for a parabola SS, the tangent at S, is the line at infinity, we get that YZ is parallel to P_1Q_1, as X is at infinity.

As before $\frac{O_1R_1}{O'Z}=\frac{O_1P_1}{O'P_1}; \quad \frac{O'Q_1}{O'R}=\frac{O_1Q_1}{O_1Y}; \quad O'Z=O_1Y.$

Multiplying these equations we get

$$\frac{O_1P_1 \times O_1Q_1}{O_1R_1}=\frac{O'P_1 \times O'R_1}{O'R}=\frac{OP \times OQ}{OR}.$$

by the first part.

Case II. Hyperbola.

In the case of a hyperbola, when the variable point, O, moves on a chord parallel to an asymptote, the theorem takes the same form as in the case of the parabola (case I, first part), viz.

$$\frac{O_1P_1 \times O_1Q_1}{O_1R}=\frac{OP \times OQ}{OR}$$

and the proof also is the same.

But the theorem apparently fails when O moves on a fixed chord and the other chord is parallel to an asymptote as pointed out in the previous note and also when both the chords are parallel to the asymptotes of a hyperbola.

Case III.

In the case of a hyperbola, the extension of the theorem holds when one chord coincides with an asymptote; in this case the theorem reduces to the form $O_1P_1 \times O_1Q_1 = OP \times OQ$.

Remembering that SS, the tangent at S, is the asymptote CS and applying Pascal's Theorem to the hexagon PQSS P_1Q_1, (Fig. 2) where S

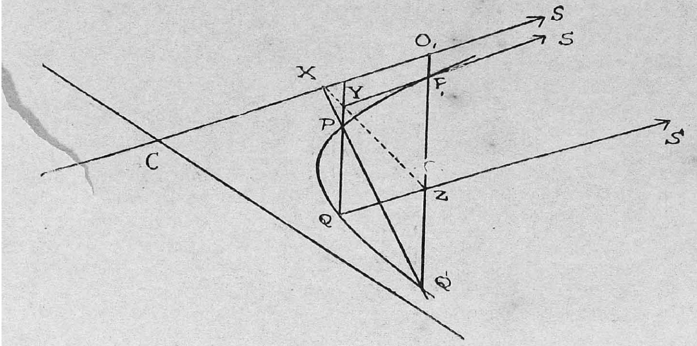

Fig. 2

is the point at infinity on the asymptote, we get that $Y=(PQ, SP_1)$, $Z=(QS, P_1Q_1)$ and X the intersection of the asymptote with PQ_1 are collinear.

As before, $\frac{OX}{O_1X}=\frac{OP}{O_1Q_1}$; $\frac{OY}{O_1Z}=\frac{OX}{O_1X}$; $O_1P_1=OY$; $O_1Z=OQ$.

Multiplying these four equations, we get the required result.

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ASTRONOMICAL NOTES FOR OCTOBER 1936

The times given hereunder are Indian Standard time.

The Sun will be moving with an apparent southeasterly motion from near the star Spica in Virgo, to Libra. From September onward, the distance from the earth to the Sun is decreasing and it will be about 92·2 million miles by the end of October. The position of the Sun on the 1st and last days of the month will be respectively R. A. 12h-29′; Decli.—3°-8′ and R. A. 14-21; Decli.—14°-4·6′.

The Moon. The phenomena of the Moon will be as follows:—

Full Moon	Oct. 1st 2h-30·8 A. M.
Last quarter	Oct. 7th.
New Moon	Oct. 15th 3h-40·4 P. M.
First quarter	Oct. 23rd.
Full Moon	Oct. 30th 11h-27·7 A. M.

The Sayana longitude on the 1st and last days of October will be respectively 13°-46′ and 52°-28·6′.

Mercury is in Virgo and will be seen in the morning sky. It is retrograde in this Constellation and attains its stationary point on the 9th. From that day onward it begins its forward motion. Its Conjunction with moon occurs on the 14th October. Its distance from the earth is increasing and it will be about 120 million miles by the end of the month. Its sayana longitude will be 187°-20′ on the 1st of October.

Venus is in Libra and will be an evening object. Its distance from the earth is decreasing and its apparent diameter is increasing. Consequently it will be brighter towards the end of October than at the beginning. Its Conjunction with Moon occurs on the 18th.

Its Sayana longitude on the 1st will be 213°-6′.

Mars will be a morning object in Leo, and its apparent motion will be easterly. A Conjunction with Nepture will occur on October 25th and with the Moon on October 12. Its Sayana longitude on the 1st October will be 152°-55′.

Jupiter will be visible in the evening sky near the Constellation Sagittarius. The distance from the earth is increasing and will be about 550 million miles by the end of the month. Conjunction with the Moon will occur on October 20. Its Sayana longitude on the 1st will be 258°-10′.

Saturn is in Aquarius, and it will be above the horizon most of the night hours. Its apparent motion among the stars is retrograde, and its distance from the earth will be about 820 million miles. Its Conjunction with the Moon occurs on October 27th. Its Sayana longitude on the 1st will be 347°-41′. During the last week of June the plane of the rings of Saturn coincided with the ecliptic.

Uranus is in Aries and its apparent motion is retrograde.

Neptune is in Leo and will be a morning object.

Comets. Two comets have been so far seen during the present year. (1936-a Peltier) and (1936-b Kaho). Their orbits have not been definitely computed.

Novae. There are two new stars visible in Hercules and Lacertae. The latter is decreasing in its brightness rapidly.

Trivandrum. H. SUBRAMANI IYER.

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43. The spiral was another geometric symbol of evolutionary force, represented in nature by the whirling of dust-storms and water-spouts, the eddying of whirlpools, or the wreaths of evaporation in water, the curling of smoke, and in Hindu allegory by the Churning of the Cosmic Ocean. It is represented in Hindu art by the coiled, or gliding snake, the antelope's and ram's horn, the Conch Shell of Vishnu and the Srivatsa curl on his breast, the Salagram stone, Narayana's navel, and the winding stalk of the Lotus.

The mystic syllable AUM was also represented by a spiral symbol which is sculptured on one of the Elephanta reliefs.

The spiral, together with the Swastika and Sauwastika and the parallel symbols of the equilateral triangles, and the three steps of Vishnu, provide the basis of innumerable intersecting patterns in Asiatic decorative art.

F. B. HAVELL, *The Ideals of Indian Art.*

SOLUTIONS TO QUESTIONS

Question 1680

(K. SATYANARAYANA) :—A pair of real isogonal conjugates with respect to a given triangle exists on a given line when and only when the line cuts a pair of opposite sides of the orthocentric tetrad formed by the in- and ex-centres one internally and one externally.

Solution by T. K. Raghavachari.

It is known that isogonal conjugates with respect to a triangle ABC are pairs of conjugate points with respect to the pencil of rectangular hyperbolas passing through the in- and ex-centres of ABC. Let I, I_1, I_2, I_3 be the in- and ex-centres respectively. Then the pairs of lines (II_1, I_2I_3), (II_2, I_1I_3), (II_3, I_1I_2) belong to this pencil. These are also the pairs of opposite sides of the orthocentric tetrad $II_1 I_2 I_3$. Let the given line cut these opposite sides in E, E′ ; F, F′ ; G, G′. If OO′ are the isogonal conjugates on the given line then

$$(OO', EE') = (OO', FF') = (OO', GG') = -1.$$

Thus O and O′ are the double points of the involution determined by EE′, FF′ and GG′.

In order that these double points may be real, it will be seen that if we take any one of the pairs of points EE′, FF′, or GG′, then the points of each of the other two pairs should lie either both within the segment connecting the first pair, or both without; and it is easily seen that this can happen when and only when the given line cuts a pair of opposite sides one internally and one externally.

Question 1681

(A. A. KRISHNASWAMY AYYANGAR).—If the F-conic of two circles degenerates, show that it is either a pair of parallel straight lines or a pair of perpendicular straight lines, and in the latter case show that the perpendicular lines meet at one of the limiting points of the coaxal system determined by the circles, the other limiting point being the centre of the degenerate ϕ conic.

Solution by B. Ramamurti.

The statements in the above problem are consequences of the degeneracy of the F conic of two conics S_1, S_2 when the latter become circles.

Referred to the common self conjugate triangle ABC of the two conics, let their equations be

$$S_1 = x^2 + y^2 + z^2 = 0 ; \qquad S_2 = ax^2 + by^2 + cz^2 = 0.$$

Then the point equation of the F conic and the line-equation of the ϕ conic are respectively

$$F = a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2 = 0$$
$$\phi = (b+c)l^2 + (c+a)m^2 + (a+b)n^2 = 0.$$

It is evident from the above equations, that the F conic is degenerate when it passes through a vertex of the common self conjugate triangle. Suppose it passes through A. Then $b+c=0$. Now

$$F = (a-b)y^2 - (a+b)z^2 = 0$$
$$\phi = (a-b)m^2 + (a+b)n^2 = 0$$

F can now be specified to be the unique pair of straight lines through A separating harmonically AB, AC, and the common chords through A. Thus we have:—

The necessary and sufficient condition that the F-conic of two conics should be degenerate is that the pairs of tangents of the two conics from a vertex of the common self conjugate triangle should separate each other harmonically. Then the F conic is the unique line pair separating harmonically the adjacent sides of the self conjugate triangle, and the common chords through that vertex. The ϕ conic, [being the reciprocal of the F conic with respect to $\sum$], reduces to a pair of points on the opposite side separating harmonically the vertices.

If S_1, S_2 are circles, the vertices of the common self conjugate triangle are A, the point at infinity on the radical axis and B, C the limiting points. Hence if the F conic passes through A, it becomes a pair of parallel straight lines, equidistant from and on either side of the radical axis. Since the pairs of tangents from A separate each other harmonically, the circles are orthogonal. If the F conic passes through a limiting point B, the common chords through B are the circular lines through B, and hence the F conic degenerates into a pair of perpendicular straight lines. Since the ϕ conic reduces to a pair of points separating harmonically C and the point at infinity A, C is the centre of the ϕ conic.

Also solved by T. K. Raghavachari and S. Narayanan.

Questions 1704 and 1728

(A. NARASINGA RAO).—Show that for any tangential pencil of quadrics,

(1) there are, in general, three quadrics through a given point

(2) the tangent planes to the 3 quadrics through a point and the 4 faces of the common self-polar tetrahedron are osculating planes of the same twisted cubic.

Solution by C. N. Srinivasiengar.

If $\sum = 0$ and $\sum' = 0$ be the tangential equations of two quadrics, $\sum + k\sum' = 0$ constitutes a tangential pencil. Writing down the point-equation of $\sum + k\sum' = 0$, k occurs in the third degree and hence follows (1). We could also obtain the result by reciprocation.

The second part can be obtained as the reciprocal theorem of a well-known result. Reciprocate with respect to any quadric. The tangent planes to the three quadrics of the tangential pencil that pass through a given point reciprocate into the points of contact of the three quadrics of a "pencil of quadric loci" touching a given plane.

The reciprocal of a twisted cubic regarded as an envelope is also a twisted cubic regarded as a locus. Hence the reciprocal theorem of (2) therefore runs thus :—In a pencil of quadric loci the points of contact of the three quadrics which touch a given plane and the four vertices of the common self-polar tetrahedron lie on the same twisted cubic. This follows from the known result: The pole of a given plane with respect to the pencil lies on a twisted cubic. [Vide Salmon: *Analytical Geometry of Three Dimensions* § 134, or Sommerville § 13·431]. That the twisted cubic passes through the vertices of the common self-polar tetrahedron is not explicitly mentioned in these books, but it is easily verified. Taking $S \equiv x^2 + y^2 + z^2 + w^2$, $S' \equiv ax^2 + by^2 + cz^2 + dw^2$, and the plane as $lx + my + nz + p = 0$, the cubic curve is given by $x = l(b-\lambda)(c-\lambda)(d-\lambda)$, $y = m(c-\lambda)(d-\lambda)(a-\lambda)$, etc. The value $\lambda = a$ gives the vertex (1, 0, 0, 0), etc. To account for these vertices geometrically, we know that there are four cones of the system, each vertex of the tetrahedron being the vertex of one cone. If we consider any of these cones, the vertex can be considered as the pole of every plane.

Question 1725

(C. T. RAJAGOPAL).—If $\sum u_n$ is a divergent series of positive terms, discuss the behaviour of the series $\sum u_n/(1 + n^s u_n)$.

Solution by S. S. Pillai

We shall show that:

(1) when $s>1$, the second series is convergent.

(2) when $s\leqslant 0$, ,, divergent.

(3) when $0<s\leqslant 1$, the second series may be convergent or divergent.

Put $u_n=1/a_n$ and $t_n=\dfrac{u_n}{1+n^s u_n}=\dfrac{1}{a_n+n^s}$

(1) Let $s>1$.

Then $t_n<1/n^s$. So $\sum t_n$ is convergent.

(2) Let $s\leqslant 0$.

Then $t_n\geqslant 1/(1+a_n)\geqslant \frac{1}{2}$, if $a_n\leqslant 1$. (*a*)

and $t_n\geqslant 1/2a_n$, if $a_n>1$. (*b*)

If $a_n\leqslant 1$ for infinity of values of n, then from (*a*) it follows that $\sum t_n$ is divergent. In the contrary case, the divergence of $\sum t_n$ follows from (*b*).

(3) $0<s\leqslant 1$.

(*a*) Let $u_n=1/n$.

Then $t_n=\dfrac{1}{n+n^s}\geqslant\dfrac{1}{2n}\cdot$ So $\sum t_n$ is divergent.

(*b*) Let

$$u_n=1/r, \text{ when } n=2^r,\ r=1,\ 2,\ 3,\ \cdots$$
$$\text{and } \quad =1/n^2 \text{ when } n\neq 2^r.$$

Then $\sum u_n$ is divergent.

Further, when $n=2^r$, $t_n=\dfrac{1}{r+n^s}<\dfrac{1}{n^s}$;

and when $n\neq 2^r$, $t_n=\dfrac{1}{n^2+n^s}<\dfrac{1}{n^2}$.

$$\text{So } \sum t_n<\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{r=1}^{\infty}\frac{1}{2^{rs}}=\frac{\pi^2}{6}+\frac{1}{2^s-1}.$$

Hence $\sum t_n$ is convergent.

Let $S=\sum 1/a_n$, $S_1=\sum 1/b_n$; $T=\sum 1/(a_n+b_n)$.

When at least one of S and S_1 is convergent, T is also convergent.

When S and S_1 are both divergent and S_1 is given, we can choose S so that T may be convergent or divergent as we please.

REVIEWS

ANDRÉ WEIL: *Arithmétique et Géométrie sur les Variétés Algébriques.*

[Actualités Scientifiques et Industrielles, no. 206, Herman, Paris 1935].

pp. 16, price 6 francs.

The pamphlet is a brief but useful memoir on what might be called "modern" algebraic geometry, the algebra being that of the post-Hilbert, or rather of the post-Emmy-Noether era. In Weil's thesis methods were given for associating each algebraic point of a curve with an ideal of an algebraic field. This distribution allowed the decomposition of any rational function on the curve into ideal factors, corresponding to the symbolic decomposition into divisors in the sense of the associated algebraic field. The method yielded valuable results in the theory of numbers, particularly in Siegel's classic memoir on diophantine approximations.

The present brochure goes further in the same direction, and in fact steps behind the scenes, as it were, of the previous theory. The field used is no longer one of algebraic number theory, but a function field (Funktionenkörper). Even so, one can set up an equivalence between homogeneous ideals and between the distributions of the ideals which is one-to-one for classes of each and gives isomorphism for operations such as products. Among some of the consequences are that the well-known results of Siegel and Mahler are independent of the method of infinite descent; "which method, if some tool more powerful could be found for its application than the theory of Abelian functions, should lead to deeper results".

D. D. K.

J. VON NEUMANN: *Charakterisierung des Spektrums eines Integraloperators*
[Actualités Scientifiques et Industrielles, no. 229. Hermann, Paris 1935]
pp. 20, price 7 francs.

This is a paper which establishes certain results in the theory of Hilbert-spaces from the point of view which has been so fruitful in the extended theory of Fourier transforms and almost periodic functions, though having its origin in the Hilbertian theory of integral equations. The transformations of a Hilbert-space which are linear in character may be described in two ways: first, by introducing normal co-ordinates (as for example, a complete set of orthogonal functions) as linear transformations in an infinite number of dimensions, characterized by an infinite matrix. The other method is essentially that of integral equations, where the transformation depends upon the kernel of the equation. While the first is clumsier to deal with directly, the second is, surprisingly enough, well-known to be more restricted in its scope in that it transfers L_2 into a sub-manifold of itself, and the identity cannot be represented in that fashion, though some highly singular operators can be.

The present work attempts to complete known methods of the second theory by inquiring into the nature of the operators which can be represented

by means of integral transforms. But rather than discuss the operator itself, its spectrum is investigated; that is to say, the author sets out as his point of view, the investigation of the conditions that a given spectrum belong to an operator of an integral transform. The work was, to some extent, initiated by Carleman, particularly in the domain of singular integral equations, and uses Weyl's lemma to transform continuous spectra into point spectra.

Four theorems are proved; all dealing with self-adjoint operators in the Hilbert space of functions measurable over $(-\infty, \infty)$ with integrable squares of absolute value. These theorems generalize Weyl's lemma; show that two self-adjoint operators with the same points of spectral concentration differ only by an additive completely continuous operator and an unitary transformation; and prove that if O is a point of concentration of the spectrum of an Hermitian hyper-maximal operator A, there exists an unitary operator U such that UAU^{-1} is an integral operator, with the converse that for each Hermitian hypermaximal integral operator, O is a concentration point of the spectrum.

D. D. K.

D. R. KAPREKAR, B.Sc., *Ten cuts in calculations,* Arya Waibhav Press, Nasik; 48 pp., Price Annas Six.

Of the ten aids to calculation given by the author, seven relate to multiplication, one is a divisibility test for 37, one treats subtraction as complementary addition while the last relates to compound interest.

The first two cuts embody the formula $(10a+b)(10a+c)=100a(a+1)+bc$ if $b+c=10$, a special case of which (cut 1) is the squaring of a number ending in 5. The third corresponds to the identity $(100a+75)(100a+25)=10000a(a+1)+1875$, which is an extension of the same principle to two digits. Cuts 4 and 5, relate to the product of two numbers $(10a+b)(10a+c)$, and applies to the cases $b+c=14, 15, 16, 17$ and to $b=c=7, 8$ or 9. The rule is: write down the unit digit in $b\times c$, then multiply the carrying figure by $(a+1)$ and write the unit digit of the product to the left of the digit already written and note again the carrying figure. Add this carrying figure to $a(a+1)$ and write this to the left of the two figures written down. I wonder whether this rule, involving three multiplications is materially simpler than actual multiplication. Cuts 8 and 9 relate to the squaring of a number ending in 25 or 75.

The divisibility test for 37, which the author claims to have discovered by himself, is to form blocks of figures by threes counting from the right. If the sum of these three-digit numbers (the one on the extreme left need not have all three digits) is divisible by 37, and in particular if it consists of the same digit repeated thrice, it is divisible by 37.

The more useful tests for divisibility by 2, 3, 5, 9 are omitted, probably because the author considers them well known. Tests for 7, 11 and 13 are much less widely known and could well have been included. The general principle underlying these tests may be stated as follows: Let $10^n \equiv r \pmod{f}$ ... (1)

Then if any number m be divided into blocks of n figures from the right, which may be represented by B_1, B_2,...etc. then

$$m = B_1 + 10^n B_2 + 10^{2n} B_3 + \dots\dots$$
$$\therefore \quad m \equiv B_1 + rB_2 + r^2 B_3 + \dots\dots \pmod f$$

and if the right is divisible by f so is the left. The most useful tests result from $r = +1$ or -1. In the former case we add the numbers in the different blocks and test this sum while in the latter case we add and subtract alternately thus forming $B_1 - B_2 + (-1)^2 B_3 + \dots\dots$

If f be prime to 10, the congruence $10^n \equiv 1 \pmod f$ has a solution $n \leq f-1$, and hence *it is possible to devise a divisibility test for every number f prime to 10 which is of the block addition type* cited above (case 1). Thus examining the factors of $10^n - 1$, we have the following table of values of n and f.

Block addition tests

n	1	2	3	4	5	6	7	8	9	10
f	9	11	37	101	41 ; 271	7; 13	239 ; 4649	73; 137	333667	9091

Thus for divisibility by 239 or by 4649 partition the number into blocks of 7 digits and test their sum. Only the primes occurring for the first time have been mentioned in the table, but the test applies to all factors f of $10^n - 1$.

On a similar examination of the factors f of $10^n + 1$ we have the following:

Block alternate addition and subtraction tests

n	1	2	3	4	5	6	7
f	11	101	7; 11; 13	73; 157	9091	9901	909091

Thus for divisibility by 7, or 11 or 13, partition into blocks of three digits and alternately add and subtract the blocks.

It may be shown that when addition only is allowed the number n need not be larger than $f-1$ while when an alternate addition and subtraction test exists, the number of digits need not exceed $(f-1)/2$. Unfortunately, such a test does not exist for every f.

The value of these tests and rules for quick multiplications depends on their simplicity and ease of application and though some of the cuts described in the book cannot come under this class, the book should stimulate an interest in numbers—one of the first steps towards acquiring a love for the subject.

A. N. RAO.

NATHAN ALTSHILLER-COURT, D.SC., *Modern Pure Solid Geometry.*

Macmillan Company, New York.

In this treatise on the synthetic geometry of the tetrahedron and the sphere are collected together theorems that "remain scattered in the various and numerous mathematical periodicals, old and new, in the form of articles and isolated propositions". The author has done a great service to the students of the subject by developing the theorems in a logical and connected order and by making them available in a compendious form. The present treatise appears to be the first of its kind in English and can be considered to be fulfilling the same purpose in three dimensional geometry as the well-known treatise on "the Geometry of the Circle" by M'clleland.

The book begins with a preliminary chapter on perspectivity in space, transversals to skew lines etc. The first three chapters are necessary for the consideration of the properties of a tetrahedron given in the next one. This latter treats of the various important points, lines and spheres connected with a tetrahedron. The student acquainted with the analytical geometry of three dimensions will appreciate the synthetic proof given in articles 320 to 328 showing that certain lines form hyperbolic groups. These are followed by instructive chapters on the properties of transversals, the oblique cone, the sphere and spherical inversion.

According to the definitions given on page 144 and 146 'conjugate lines' is used as synonymous with 'polar lines' for a sphere and 'self conjugate tetrahedron' with self-polar tetrahedron. But the usage in Analytical conics hitherto seems to have been different. In Baker's "Principles of Geometry" Vol. 3, Page 34 or in Sommerville's "Analytical Geometry" Page 130 two lines are defined to be conjugate for a quadric when one intersects the polar of the other; and with this definition, a self conjugate tetrahedron for a quadric is the one having its pairs of opposite sides conjugate, and *not polar*, for the quadric. Nothing is gained by this departure from accepted terminology.

The last chapter of the book, "Recent Geometry of the Tetrahedron" will be read with special interest by one who has perused Casey's Analytical Conics or his Sequel to Euclid. That there are bound to be many points of similarity between the geometry of the triangle and of the tetrahedron will be readily accepted by any imaginative student of geometry. The author is to be congratulated therefore in bringing out the essential points of similarity and in placing before the student sufficient matter to enable him to investigate for himself many other interesting properties of tetrahedrons not yet noticed. The theory of tetrahedral poles and polars, of antiparallels, and isogonal conjugate points, the properties of the Monge point in an orthocentric tetrahedron; the isodynamic tetrahedron with its Lemoyne point and Neuberg spheres—all these remind the reader of the analogous results in triangle geometry.

Numerous graded examples are given at the end of every section and chapter, and an extensive Bibliography and index enhance the value of the book. It is to be hoped that this book will give the necessary impetus to the study of this somewhat neglected branch of geometry.

T. R. R. SASTRI.

DAVID EUGENE SMITH: *Portraits of Eminent Mathematicians,*
Portfolio I: Scripta Mathematica, New York, 1936.

Scripta Mathematica, a Journal published by the Yeshiva College, New York, has within the short period of 4 years achieved not only a reputation as an ably conducted and very readable quarterly on the Philosophy and History of Mathematics but has also to its credit a number of interesting booklets of Mathematical essays which form the *Scripta Mathematica Library.*

Not content with these services, it has now published a portfolio of Portraits of 12 eminent Mathematicians viz.: Archimedes (reproduced from a mosaic found in Pompeii depicting the death of Archimedes at the hands of a Roman soldier), Copernicus, Viète, Galileo, Napier, Descartes, Newton, Leibnitz, Lagrange, Gauss, Lobatchewsky and Sylvester. Each portrait is printed on thick art paper (roughly $10'' \times 13\frac{1}{2}''$) and is accompanied by a folder containing biographical material, and in most cases a facsimile of their handwriting and of the title pages of their most important publication. The whole is neatly encased in a cardboard carton bearing an illustration of the Mathematical themes in design which have been appearing in the Scripta.

No one doubts the value of a historical and human background in the teaching of an abstract subject like Mathematics and these portraits should help greatly—especially if framed and hung in class rooms—in making students feel that they know something of the men who have helped in the making of the subject.

Let us hope that this portrait gallery will be extended so as to include all prominent Mathematicians who are nearer our own times including those whom we have the privilege of having in our midst to-day.

A. N. RAO.

ANNOUNCEMENTS AND NEWS

The next number of THE MATHEMATICS STUDENT will be a special issue dedicated to the memory of the late V. Ramaswami Aiyar, Founder of the Indian Mathematical Society.

The following persons have been admitted as members of the society:—

J. N. Mitra Esq., M.A., Professor, Ramjas College, Delhi.

Jeo Kishan Dayal (Miss), M.A., Professor, Lahore College of Women, Lahore.

G. L. Chandratreya Esq., B.Sc. (Bom.), M.A., B.A. (Cantab.), Professor, Fergusson College, Poona.

K. Mahadeva Iyer Esq., B.A., Accountant, Collector's Office, Chittoor.

V. Rangachariar Esq., M.Sc., Science College, Patna.

Ratnakar Sharangi Esq., M.Sc., Deputy Magistrate, Puri.

The 24th Annual Meeting of the Indian Science Congress will be held at Hyderabad (Deccan) between 2nd and 8th January, 1937 under the patronage of H. E. H. the Nizam of Hyderabad. The General President is Rao Bahadur

T. S. Venkataraman, Imperial Sugarcane Expert, Coimbatore, while the sectional President for Mathematics and Physics will be Dr. S. Datta, Professor of Physics, Presidency College, Calcutta.

We understand that Dr. Royds of the Kodaikanal observatory has submitted detailed plans for the construction of an observatory to be located in a turret of His Highness the Maharaja's Palace on the Chamundi Hills, Mysore, 3,300 feet above sea level. As at present planned it will consist of an equitorial telescope of 6 inches aperture fixed on a revolving dome, and a transit instrument of 4 inches.

At the forthcoming transit of Mercury in 1937, the planet will pass along the limb of the Sun cutting it by only a few seconds of arc. Such a transit is expected to provide a rare opportunity of seeking the possible existence of a refraction ring indicating the presence of planetary atmosphere.

The path of totality at the recent solar eclipse of June 19th ran from the south of Greece through Asiatic Russia and Japan into the Pacific. The Eclipse expeditions sent to the Aegian Sea, and Omsk (Russia) met with good weather which, it appears, did not favour the expeditions to Hokkiado (Japan).

A nova—Nova Lacertae—was observed in the region between Cepheus and Lacertae on June 18th 1936, when it was estimated to be of magnitude 3·5. By June 20th the magnitude had risen sharply to 2·2 but afterwards declined to about 4·5 by June 29th and to 6·4 on July 24 and to 7·8 by 19th of August.

The Paris Academy of Sciences has awarded the Francouer Prize to André Weil, who was for sometime Professor at Aligarh, for his work on Algebraic Varieties.

The gold medal of the Royal Astronomical Society has been awarded to Prof. Kimura, Director of the International Latitude Observatory at Mizusawa in Japan. His principal discovery relates to the existence of a small variation that is constant with respect to the longitude of the station but varies with the latitude. This is the first time the medal has been awarded to one outside Europe or North America.

M. Zia-ud-din, M.A. (Aligarh), Professor of Mathematics, S. E. College, Bawalpore, returned to India in August 1936 after taking the Ph. D. degree of the University of Wales. The subject of his thesis was "Symmetric Functions."

B. Ramamurthi, M.A., of the Annamalai University has been awarded the D. Sc. Degree of the University of Madras, his thesis being entitled "Some contributions to the Geometry of the Rational Norm Curve and the Rational Normal Surface."

The *Deutsche Mathematik* a new Mathematical Journal is being received in exchange by the Society.

We regret to announce the death of Dewan Bahadur R. Ramachandra Rao, a former President (1915-17) of the Indian Mathematical Society and one who materially helped S. Ramanujan in his early struggles against poverty and neglect.

OBITUARY

V. B. NAICK

Vishwanath Balavant Naick was born in 1880. After matriculating from the Poona High School, he studied in the Maharashtra College and then in the Fergusson College whence he appeared for the B.A. degree examination in 1901, and four years later took the Master's degree. He belonged to the very first batch of students of Mathematics who studied under Dr. Paranjpye after his return from England. He was considered a student of high ability and character, and was a Dixina Fellow for two years (1902, and 1903). As the leading student of the year, the honour of drafting and reading the farewell address to Gokhale on behalf of students (September 1902) fell to him. Gokhale must have formed a very favourable opinion of him, for we learn that he was invited to join the Servants of India Society. A scholar's life had, however, greater attractions for him and he decided to join the Deccan Education Society.

Clear thinking and accurate presentation were the outstanding merits in Prof. Naik's teaching. He possessed the art of making his lessons interesting and instructive to the scholar and the back-bencher alike. He lectured in Astronomy (his favourite subject), Hydrostatics and Algebra. Prof. Naik published two books, one on Solid Geometry in collaboration with Prof. Bondale and the other a "College Algebra" in collaboration with Prof. Apte.

Prof. Naik took a keen interest in matters educational and was superintendent of the Navin Marathi Shala and the New Poona English school. He took a lead in organising the Siksha Vichar Mandal and collaborated with Prof. M. R. Paranjpye in starting the journal—The Progress of Education. He was one of the Foundation members of the Indian Mathematical Club and served firstly as assistant Librarian and later as Honorary Librarian, and continued to serve the Society till his death in February 1936.

Prof. Naik will be remembered as the very soul of gentility and for his unswerving loyalty to truth and abhorrence of wrong and injustice. His sense of duty was exalted and his capacity for work immense, in spite of a weak constitution. He was religious and orthodox in the best sense of the terms. He liked to interpret and follow religion in his own way, but he was always sympathetic towards the differing views and practices of others.

[Adapted from *The Fergusson College Magazine*]

QUESTIONS FOR SOLUTION

1730. (V. THEBAULT, Le Mans, France)—

With the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 taken each once, form a perfect square whose root is of the form $a\,a\,b\,c\,c$.

1731. (A. WEILAND, Moskow):

We have:

$$1^2 + 3^0 = 2^1, \qquad 2 + 0 + 1 = 1 + 2$$
$$2^2 + 5^1 = 3^2, \qquad 2 + 1 + 2 = 2 + 3$$
$$3^5 + 10^2 = 7^3, \qquad 5 + 2 + 3 = 3 + 7$$

Are the equations

$$x^z + (x + y)^u = y^t$$
$$z + u + t = x + y$$

solvable?

What is the general solution?

1732. (ALFRED MOESSNER, Nurnberg, Germany)

What is the general solution of the equations

$$A^n + B^n + C^n = D^n + E^n + F^n \qquad (n = 2,\ 4)$$

when we have also

$$A^2 + B^2 = E^2; \qquad D^2 + F^2 = C^2$$

Example $20^2 + 21^2 + 37^2 = 12^2 + 29^2 + 35^2$

$$20^4 + 21^4 + 37^4 = 12^4 + 29^4 + 35^4$$

while we have also

$$20^2 + 21^2 = 29^2$$
$$12^2 + 35^2 = 37^2.$$

1733. (A. A. KRISHNASWAMI AYYANGAR)—

A variable tangent to a conic meets a fixed tangent in P and a given parallel chord in Q. S is a fixed point in the plane of the conic. If QR is drawn parallel to PS, show that the envelope of QR is a parabola.

If this parabola and the given conic have a common focus, determine its position.

1734. (C. N. SRINIVASIENGAR)—The equations of two conics with a common focus are written

$$S \equiv \frac{l}{r} - 1 - e \cos \theta = 0 ;$$

$$S' \equiv \frac{l'}{r} - 1 - e' \cos (\theta - \alpha) = 0.$$

The equation $l' S = l S'$ represents a pair of lines through the origin, but these lines are not common chords of the conics.

Account for this and show that this peculiarity is generally characteristic of curves defined by polar equations.

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44. Accordingly, from this ocean of SARASANGRAHA, which is filled with the water of terminology and has the (eight) arithmetical operations for its bank; which (again) is full of the bold rolling fish represented by the operations relating to fractions, and is characterised by the great crocodile represented by the chapter of miscellaneous examples; which (again) is possessed of the waves represented by the chapter on the rule of three, and is variegated in splendour through the lustre of the gems represented by the excellent language relating to the chapter on mixed problems; and which (again) possesses the extensive bottom represented by the chapter on area-problems, and has the sands represented by the chapter on cubic contents of excavations; and wherein (finally) shines forth the advancing tide represented by the chapter on shadows, which is related to the department of practical calculation in astronomy—(from this ocean) arithmeticians possessing the necessary qualifications in abundance will, through the instrumentality of calculation, obtain such pure gems as they desire.

MAHAVIRACHARYA in *Ganitasarasangraha* (about 850 A.D.)

tr. by M. RANGACHARYA.

45. "Now is not this very fine? Mathematicians that find out, settle, and do all the business must content themselves with being nothing but dry calculators and drudges, and another that does nothing but pretend and grasp at all things must carry away all the invention."

NEWTON'S LETTER TO HALLEY IN 1686.

190021
Bm211,N33MS

The Indian Mathematical Society

(*Founded in 1907 for the Advancement of Mathematical Study and Research in India.*)

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All communications intended for publication should be sent to the Editor, [illegible]SINGA RAO, Annamalai University, Annamalainagar, South India. Communi-[illegible]ing to the receipt of the Journal, changes of address, and the purchase of [illegible] back numbers of this journal should be sent to the Assistant Secretary, [illegible]DEVA IYER, Assistant Professor of Mathematics, Presidency College, Madras.

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